

# PFAFFIAN ORIENTATIONS, FLAT EMBEDDINGS, AND STEINBERG'S CONJECTURE

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# PFAFFIAN ORIENTATIONS, FLAT EMBEDDINGS, AND STEINBERG'S CONJECTURE

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*To Krista*

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## SUMMARY

The first result of this thesis is a partial result in the direction of Steinberg's Conjecture. Steinberg's Conjecture states that any planar graph without cycles of length four or five is three colorable. Borodin, Glebov, Montassier, and Raspaud showed that planar graphs without cycles of length four, five, or seven are three colorable and Borodin and Glebov showed that planar graphs without five cycles or triangles at distance at most two apart are three colorable. We prove a statement that implies the first of these theorems and is incomparable with the second: that any planar graph with no cycles of length four through six or cycles of length seven with incident triangles distance exactly two apart are three colorable.

The third and fourth chapters of this thesis are concerned with the study of Pfaffian orientations. A theorem proved by William McCuaig and, independently, Neil Robertson, Paul Seymour, and Robin Thomas provides a good characterization for whether or not a bipartite graph has a Pfaffian orientation as well as a polynomial time algorithm for that problem. We reprove this characterization and provide a new algorithm for this problem. In Chapter 3, we generalize a preliminary result needed to reprove this theorem. Specifically, we show that any internally 4-connected, non-planar bipartite graph contains a subdivision of  $K_{3,3}$  in which each path has odd length. In Chapter 4, we make use of this result to provide a much shorter proof using elementary methods of this characterization.

In the fifth and sixth chapters we investigate flat embeddings. A piecewise-linear embedding of a graph in 3-space is *flat* if every cycle of the graph bounds a disk disjoint from the rest of the graph. We provide a structural theorem for flat embeddings that indicates how to build them from small pieces in Chapter 5. In Chapter 6, we present

a class of flat graphs that are highly non-planar in the sense that, for any fixed  $k$ , there are an infinite number of members of the class such that deleting  $k$  vertices leaves the graph non-planar.

# CHAPTER I

## INTRODUCTION

In this chapter, we present the basic terminology and definitions that serve as the foundation for the work to follow as well as some motivation for the various results presented in this paper. In Section 1.1 we present a brief overview of the basic definitions in graph theory. In Section 1.2 we provide some motivation and history for coloring graphs in the plane. In Section 1.3 we provide an introduction to matching theory and the topic of Pfaffian orientations. Finally, in Section 1.4, we provide context for questions regarding embedding graphs in three dimensions.

### 1.1 *Graph Theory*

We follow the exposition of Bondy and Murty from [7]. A *graph* is an ordered triple  $(V(G), E(G), \phi_G)$  in which  $V(G)$  is a non-empty set of *vertices*,  $E(G)$  is a set disjoint from  $V(G)$  of *edges*, and  $\phi_G$  is an *incidence function* that maps each edge onto a pair of vertices. If  $e$  is an edge with  $\phi_G(e) = uv$  then we say that  $u$  and  $v$  are the *ends* of  $e$  and  $e$  *joins*  $u$  and  $v$ . If there exists an edge  $e$  such that  $\phi_G(e) = uv$  then we say that  $u$  and  $v$  are adjacent to one another. A *loop* in a graph is an edge  $e$  with both ends the same. *Parallel edges* (or *multiple edges*) are edges  $e, f$  with the same ends. Graphs that contain no loops or parallel edges are *simple*. When we need to distinguish those graphs that do allow loops and parallel edges, we refer to them as *multigraphs*. When it is clear from the context whether we are dealing with simple graphs or multigraphs we often omit the distinction and refer to them as just graphs. In order to depict graphs, we often represent them with vertices drawn as dots and edges as lines between them.

For a graph  $G = (V, E, \phi)$ , let  $V' \subseteq V, E' \subseteq E$ ,  $\phi'$  be the restriction of  $\phi$  to  $E'$

and, for every edge in  $E'$ , both ends are in  $V'$ . Then we say that  $G' = (V', E', \phi')$  is a subgraph of  $G$ . A subgraph  $(V', E', \phi')$  is said to be *induced* if  $E'$  contains every edge of  $E$  with both ends in  $V'$ . If  $u$  and  $v$  are vertices of  $G$ , when we *identify*  $u$  and  $v$ , we remove  $u$  and  $v$  from  $G$ , add a new vertex  $w$ , and, for each edge  $e$  with  $\phi(e) = (u, x)$  or  $\phi(e) = (v, x)$ ,  $x \in V$ , we set  $\phi(e) = (w, x)$  instead. Edge *contraction* is the act of identifying two adjacent vertices and deleting the edge between them. We say that a graph  $H$  is a *minor* of another graph  $G$  if we can form  $H$  from  $G$  by a sequence of deletions of vertices, deletions of edges, and contractions of edges. We say that a graph  $H$  is a *subdivision* of a graph  $G$  if  $H$  can be formed from  $G$  by repeatedly replacing edges of  $G$  with paths that are vertex disjoint from  $G$  except for their ends. Finally, we say that two graphs  $G$  and  $H$  are *isomorphic* if there exist bijections  $\phi : V(G) \rightarrow V(H)$  and  $\rho : E(G) \rightarrow E(H)$  such that if the ends of  $e \in E(G)$  are  $u$  and  $v$ , then the ends of  $\rho(e)$  are  $\phi(u)$  and  $\phi(v)$ . We say that  $u$  and  $v$  are neighbors and we use the notation  $N(u)$  to be the set of neighbors of  $u$ . We define  $\deg(u)$  to be the number of edges incident with  $u$  with loops counted twice and extend the definition of  $N$  to sets of vertices  $X$ , so that  $N(X)$  is the set of vertices not in  $X$  with at least one neighbor in  $X$ .

We often discuss subpaths and their unions so we introduce notation to facilitate this. Let  $P$  be a path and  $u$  and  $v$  vertices of  $P$ . Then we refer to the subpath of  $P$  between  $u$  and  $v$  (inclusive) as  $uPv$ . For convenience, when it does not create ambiguity, we also refer to  $V(uPv)$  as  $uPv$ . Similarly, let  $P_1, \dots, P_k$  be paths and  $x_0, x_1, \dots, x_k$  vertices with  $x_{i-1}$  and  $x_i$  on  $P_i$  for  $1 \leq i \leq k$ . Then  $x_0P_1x_1P_2x_2 \dots x_kP_k = \bigcup_{i=1}^k x_{i-1}P_ix_i$ .

We will use a number of other terms throughout this paper and generally use common terminology without explicit definition. For a formal introduction to these terms, we refer readers to any introductory text in graph theory, for example the book by Diestel [13].

## 1.2 Planar Coloring

A coloring of a graph is a mapping  $\phi : V(G) \rightarrow \mathbb{Z}$  such that if  $u$  and  $v$  are adjacent,  $\phi(u) \neq \phi(v)$ . The chromatic number of a graph is the smallest value  $\chi$  such that  $\chi$  is the size of the range of some coloring of  $G$ . An early question in coloring theory arose from cartographers who, apocryphally, noticed that their maps could be colored using only a small number of colors with the property that countries sharing a border had different colors. In modern times, graph coloring questions have relevance in different fields, for instance in register allocation for compilers. While these modern applications are concerned with general graphs, the classical question is mostly concerned *planar* graphs, those that can be drawn on the plane without crossings, beginning with the famous Four Color Conjecture (now Theorem):

**Theorem 1.2.1.** *Every planar graph has chromatic number at most 4.*

An important open question for over a century, the four color theorem was settled in 1976 by Appel and Haken [2, 3]. This theorem does not entirely settle the question of planar graph coloring: specifically, it is reasonable to ask the question “When does a planar graph have chromatic number strictly less than 4”? It is well known that any graph has chromatic number 2 if and only if it contains no odd cycles, but determining whether planar graphs are three-colorable is fundamentally difficult. Garey, Johnson, and Stockmeyer [16] proved that determining whether a planar graph is three colorable is NP-complete, so any characterization that leads to a polynomial-time algorithm is unlikely. There are a number of partial results, however, for certain types of planar graphs. One such classical result is Grotzsch’s Theorem:

**Theorem 1.2.2.** *Triangle-free planar graphs are three-colorable.*

Various extensions of this theorem exist. For example, Thomassen showed in [48] that planar graphs with no three or four cycles are 3-list colorable. In a list coloring, each vertex is assigned a list of colors of a particular length and the color assigned to

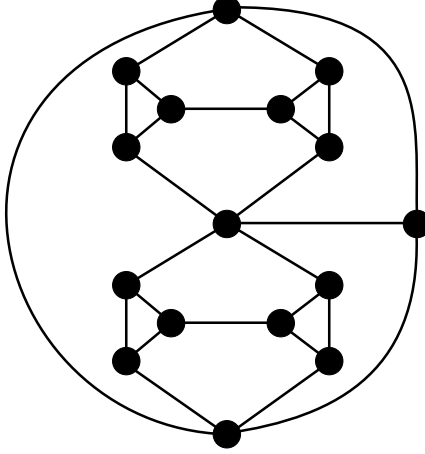


Figure 1.1: Planar graph with chromatic number 4 and no 4-cycles

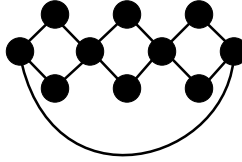


Figure 1.2: Planar graph with chromatic number 4 and no 5-cycles

each vertex must come from that list. Other extensions look at allowing some small number of triangles or allowing triangles so long as they are sufficiently far apart.

A natural next question is “What happens when we allow triangles”? Richard Steinberg proposed the following conjecture:

**Conjecture 1.2.3.** Planar graphs with no cycles of length four or five are three-colorable

We note first that this conjecture is best possible in the sense that there are planar graphs with no cycles of length 4 that are not three colorable (for instance that in Figure 1.1 and those with no cycles of length 5, such as that depicted in Figure 1.2.

While this conjecture remains unsolved, there has been substantial progress made on various fronts. Paul Erdős suggested in 1991 that instead of trying to only exclude 4 and 5 cycles, we instead exclude cycles of lengths 4 through  $k$ . Abbott and Zhou [1] showed that planar graphs excluding cycles of length four through eleven are three

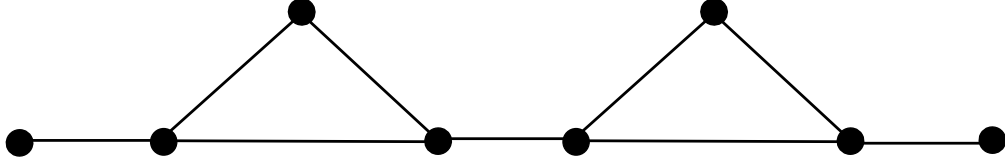


Figure 1.3: A Tetrad

colorable in 1991. In 1995, Sanders and Zhao [44] improved this to cycles of length four through nine. This approach has so far culminated in the following theorem

**Theorem 1.2.4.** *(Borodin et al) [10] Every planar graph without cycles of length from 4 to 7 is 3-colorable.*

These coloring arguments, like the proof of the four color theorem, tend to use a *reduction-discharging* approach. The first half of the arguments provide a list of subgraphs that are not present in a minimal counterexample. Such structures have the property that after identifying a pair of vertices and possibly deleting others, the resulting graph is smaller and therefore colorable. After unidentifying the vertices, the resulting coloring extends to the remainder of the graph. The *tetrad* depicted in figure 1.3 is one common reducible configuration first discussed by Borodin in [10].

The second half of such arguments is the discharging step, based on Euler's formula:

**Theorem 1.2.5.** *In a connected plane graph,  $V + F = E + 2$  where  $V$  is the number of vertices,  $F$  the number of faces, and  $E$  the number of edges.*

In our context, this is often rewritten as:

**Theorem 1.2.6.** *In a connected plane graph,  $\sum_{v \in V(G)} (\deg(v) - 4) + \sum_{f \in F(G)} (|f| - 4) = -8$  where, for a face  $f$ ,  $|f|$  is the length of the walk bounding  $f$*

In a discharging argument, we assign charge to each vertex and to each face, in our case equal to the degree of the vertex minus four or the size of the face minus



four. The total charge in the graph is then  $-8$ . We then apply a set of rules by which charge moves between faces and vertices so that each interior face and vertex has non-negative charge. By then bounding the charge on the exterior face away from  $-8$ , we then reach a contradiction.

A different type of approach than the Erdős method is to exclude 5 and 7 cycles (so not cycles of size four or six) and to look at the minimum distance between two triangles. Baogang Xu showed [53] in 2006 that any such graph without intersecting triangles is three colorable. If we choose not to exclude 7 cycles (so only exclude five cycles), Borodin and others [11] showed in 2003 that any planar graph without five cycles and without triangles at a distance less than four is three colorable, a result later strengthened by Xu to only exclude triangles at a distance less than 3. A recent paper by Borodin and Glebov [8] improves this further to

**Theorem 1.2.7.** *(Borodin and Glebov) [8] Every planar graph without cycles of length 5 and with the minimum distance between triangles at least 2 is three colorable.*

In Chapter 2, we will present a strengthening that combines some of these results: we forbid cycles of length 4, 5, and 6 and allow cycles of length 7 so long as they do not have incident triangles too close together. The main thrust of Chapter 2 is to formalize the preceding statement and to provide a proof which includes techniques that may be of use in further investigations of Steinberg’s Conjecture.

### ***1.3 Matchings and Pfaffian Orientations***

In a graph  $G$ , a *matching* is a set of edges with the property that no two share an end. A *perfect matching* is a matching in which each vertex is incident with one of the edges of the matching. Matchings have significant historical importance as well as substantial interest in the literature. For example, Hall’s celebrated Marriage Theorem characterizes those bipartite graphs that contain perfect matchings:

**Theorem 1.3.1** (Hall). *Let  $G$  be a bipartite graph with bipartition  $(A, B)$ . Then  $G$  has a complete matching from  $A$  to  $B$  if and only if for every  $X \subseteq A$ ,  $|N(X)| \geq |X|$ .*

In general graphs, the characterization is slightly more complex, but also well-known. In the following, let  $o(X)$  where  $X$  is a set of vertices be the number of odd components of  $G$  after deleting  $X$ .

**Theorem 1.3.2** (Tutte). *Let  $G$  be a graph. Then  $G$  has a perfect matching if and only if for every  $X \subseteq V(G)$ ,  $o(X) \leq |X|$ .*

We note that the problem of finding a perfect matching in a graph algorithmically also has a classical solution in the Blossom Algorithm of Edmonds [14].

Before we turn our attention toward the main thrust of this section, we mention one natural definition and extension. We say that a graph is *k-extendable* if for every set of disjoint edges of size  $k$ , there exists a perfect matching containing that set. A connected 1-extendable graph is *matching-covered*. This leads to the following straightforward extension of Hall's Theorem:

**Theorem 1.3.3.** *Let  $G$  be a connected bipartite graph with bipartition  $(A, B)$  with  $k \geq 0$ . Then  $G$  is  $k$ -extendable if and only if for every  $X \subseteq A$ ,  $N(X) = B$  or  $|N(X)| \geq |X| + k$ .*

Since the question of finding a single perfect matching in a graph has elegant classical solutions, we turn our attention instead to the problem of counting the number of perfect matchings in a particular graph. Here there is little hope of finding an algorithmically good solution to the problem; indeed work by Valiant has shown that counting the number of perfect matchings in a general graph is  $\#P$ -Complete [49]. Work by Kasteleyn [21], however gives us hope as to how to proceed by defining a *Pfaffian orientation* of a graph:

**Definition.** Let  $G$  be a directed graph. Then  $G$  is *Pfaffian* if for every even cycle in the underlying undirected graph,  $C$  such that  $G - V(C)$  contains a perfect matching,

the number of edges directed in either direction of the cycle is odd. If  $G$  is an undirected graph, we say that  $G$  is *Pfaffian* if there is an orientation of the edges such that the resulting directed graph is Pfaffian. Such an orientation is called a *Pfaffian orientation*.

Pfaffian orientations are of interest due to two classical theorems of Kasteleyn:

**Theorem 1.3.4.** *There is a polynomial-time algorithm to count the number of perfect matchings in a graph given a Pfaffian orientation*

**Theorem 1.3.5.** *Every planar graph is Pfaffian*

The first of these theorems arises from a natural linear-algebraic interpretation of the Pfaffian of a graph. Specifically, we define the *skew-adjacency matrix* of a directed graph  $G$  with vertex set  $\{1, 2, \dots, n\}$  to be the matrix with values  $a_{ij}$  where

$$a_{ij} = \begin{cases} 1 & ij \in E(G) \\ -1 & ji \in E(G) \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

Let  $A$  be a  $2n \times 2n$  matrix. Let  $P$  be a partition of the elements of  $[1, 2, \dots, 2n]$  into unordered pairs  $(i_1, j_1), (i_2, j_2), \dots, (i_n, j_n)$  and define the permutation

$$\pi(P) = \begin{bmatrix} 1 & 2 & 3 & \dots & 2n-1 & 2n \\ i_1 & j_1 & i_2 & \dots & i_n & j_n \end{bmatrix} \quad (2)$$

Then the Pfaffian of  $A$  is

$$\text{Pf}(A) = \sum_P \text{sgn}(\pi(P)) a_{i_1 j_1} a_{i_2 j_2} \dots a_{i_n j_n} \quad (3)$$

Where the sum above is taken over all possible partitions  $P$ . In the case where  $A$  is the skew-adjacency matrix of a Pfaffian orientation of a graph, the absolute value of

the pfaffian of  $A$  is the number of perfect matchings of  $G$ . This can then be computed easily using a fact from linear algebra for skew-symmetric matrices:

$$\det(A) = (\text{Pf}(A))^2 \tag{4}$$

Since the matrix  $A$  is easy to write down and the determinant is polynomial-time to compute, this immediately gives a polynomial time algorithm to count the number of perfect matchings in a Pfaffian graph.

The problem of determining whether or not a graph has a Pfaffian orientation is wide open in general. It is, however, solved for a fair number of classes of graphs and there exist several other non-polynomial time computable characterizations for it.

Note first that if we wish to determine whether or not  $G$  is Pfaffian, we may delete any edge of  $G$  that is not in a perfect matching since it has no effect on the outcome. Similarly we may assume that  $G$  is connected since  $G$  is Pfaffian if and only if each of its components is. We introduce here two classes of matching-covered graphs that are of particular interest. Bricks are 3-connected matching covered graphs such that  $G \setminus u \setminus v$  has a perfect matching for every two distinct vertices  $u$  and  $v$ . Braces are matching covered bipartite graphs in which each matching of size 2 can be extended to a perfect matching.

By a theorem of Lovasz and Plummer [30], any matching covered graph can be decomposed in a way unique up to edge multiplicities into bricks and braces, referred to as the bricks and braces of  $G$ , in polynomial time. Vazirani and Yannakakis [51] made use of this decomposition to prove that

**Theorem 1.3.6.**  *$G$  is Pfaffian if and only if all its bricks and braces are Pfaffian.*

This reduces the question of determining whether a graph has a Pfaffian orientation to that of determining whether or not bricks and braces have a Pfaffian orientation. While this remains open for bricks, there is a characterization of Pfaffian

bipartite graphs due to Little that requires a small number of preliminaries. Given a vertex of degree 2, we refer to the process of contracting both incident edges as *bicontraction*. Then let  $G$  be a matching covered graph. We say that  $H$  is a matching minor of  $G$  if  $G$  contains a subgraph  $H'$  so that  $G - V(H')$  contains a perfect matching and  $H$  can be obtained from  $H'$  by repeatedly bicontracting edges. This leads to Little's elegant result [29]:

**Theorem 1.3.7.** *Let  $G$  be a bipartite graph. Then  $G$  is Pfaffian if and only if it does not contain  $K_{3,3}$  as a matching minor.*

The disadvantage to this theorem is that it does not immediately give rise to a polynomial time algorithm. Instead, a result of Robertson, Seymour, and Thomas [42] provides an algorithm for determining whether a brace has a Pfaffian orientation which relies on a theorem also independently proven by McCuaig [31]:

**Theorem 1.3.8.** *A brace has a Pfaffian orientation if and only if it is isomorphic to the Heawood graph, or if it can be obtained from planar braces by repeated application of the trisum operation.*

Here, the trisum operation is a way of gluing together three different graphs found in [42]. Specifically, let  $G_0$  be a graph and  $C$  a cycle of  $G_0$  of length 4 such that  $G_0 \setminus V(C)$  contains a perfect matching. Let  $G_1, G_2, G_3$  be subgraphs of  $G_0$  such that  $G_1 \cup G_2 \cup G_3 = G_0$  and for distinct  $i, j \in \{1, 2, 3\}$ ,  $G_i \cap G_j = C$  and  $V(G_i) - V(C) \neq \emptyset$ . Let  $G$  be obtained from  $G_0$  by deleting some (possibly none) of the edges of  $C$ . Then  $G$  is a *trisum* of  $G_1, G_2$  and  $G_3$ .

The underlying theorem in this case was actually:

**Theorem 1.3.9.** *Let  $G$  be a nonplanar brace. Then  $G$  contains one of  $K_{3,3}$ , the Heawood graph, or Rotunda as a matching minor.*

We note that in addition to providing a way to test whether a brace has a Pfaffian orientation, this theorem, using an algorithm of Kasteleyn for planar graphs also

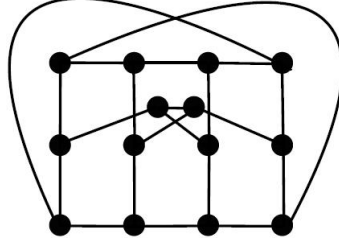


Figure 1.4: The Heawood Graph

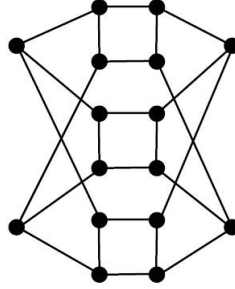


Figure 1.5: Rotunda

provides a way to *find* a Pfaffian orientation in polynomial time. This result has broader impact as well, solving several equivalent problems of interest beyond graph theory. For example, a longstanding problem in economics is the question of whether a matrix whose entries are represented only by signs is non-singular [43]. The Pólya permanent problem which asks when can the permanent of a  $0-1$  matrix be computed by computing the determinant of a matrix related by replacing 1's and  $-1$ 's in the original matrix. Both of these questions (and others) are equivalent to the problem of finding Pfaffian orientations in bipartite graphs [42] .

While this theorem provides a nice solution to a long open problem, the proof requires substantial effort. We provide in Chapter 4 a significantly shorter proof using elementary techniques. In Chapter 3, we prove a theorem for finding odd subdivisions of  $K_{3,3}$  in well-connected bipartite graphs; a theorem of interest both for its use in Chapter 4 in regards to matching theory and on its own merits.

## 1.4 *Embeddings*

An *embedding* of a graph  $G$  in a topological space  $\Sigma$  is a representation of  $G$  in which vertices of  $G$  are mapped to points of  $\Sigma$  and edges of  $G$  are mapped to arcs of  $\Sigma$  (images of  $[0, 1]$  under homeomorphism) in such a way that for each edge  $e \in E(G)$ ,  $e = uv$ , the endpoints of the arc associated to  $e$  are the points associated with  $u$  and  $v$ . Further, no two arcs intersect except at their ends. In our context we will restrict embeddings to be *piece-wise linear*, which means that each arc is the union of a finite number of straight lines, disjoint from one another except at their ends.

We are often interested in the question, "Given a topological space, which graphs embed in it?". The simplest of these questions is for planar graphs, those that embed on the plane. A classical theorem of Kuratowski [28] gives an elegant solution:

**Theorem 1.4.1.** *A graph is planar if and only if it does not contain  $K_5$  or  $K_{3,3}$  as a minor.*

It is easy to see that if  $H$  does not embed in  $\Sigma$  and  $G$  contains  $H$  as a minor then  $G$  does not embed in  $\Sigma$  and similarly if  $G$  embeds in  $\Sigma$  and  $H$  is a minor of  $G$ , then  $H$  also embeds in  $\Sigma$ . The property of embeddings is therefore minor-closed, so a deep theorem of Robertson and Seymour [38] tells us that

**Theorem 1.4.2.** *For any minor closed and isomorphism-closed set  $g$  of graphs, there is a finite subset  $H_1, H_2, \dots, H_k$  such that a graph does not belong to  $g$  if and only if it has a minor isomorphic to some  $H_i$ ,  $1 \leq i \leq k$ .*

Therefore there is a Kuratowski-like minor exclusion theorem for every topological space. This theorem is non-constructive, however, so the list of graphs that must be excluded is known for only two topological surfaces: the plane and the projective plane. In the projective plane, there are 35 minor-minimal graphs, a theorem proved by Archdeacon [4] from a list originally compiled by Glover, Huneke, and Wang [17]. In the algorithmic setting, there are several linear-time algorithms to test whether a

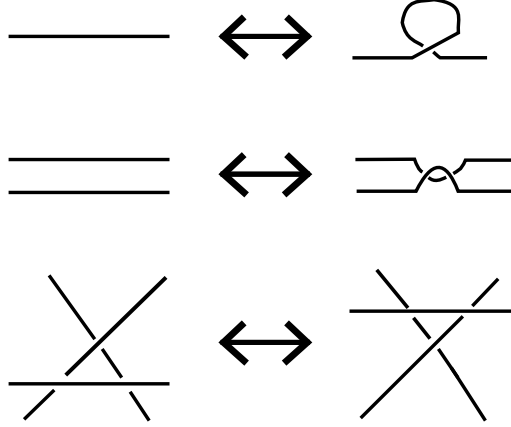


Figure 1.6: The Reidemeister Moves

given graph is planar and to find a planar embedding. Work by Mohar [33] extends this to other surfaces, giving a linear time algorithm to test whether a graph is embeddable on a fixed surface and then finding such an embedding if it exists. If the surface is part of the input, however, this problem becomes NP-hard.

While graph embeddings on surfaces are reasonably well understood, we turn our attention instead to embeddings of graphs in 3-dimensions. Every graph has an embedding in  $\mathbb{S}^3$ , the 3-dimensional sphere. We can see this by taking the book embedding of a graph  $G$ : embed each of the vertices of  $G$  on a line and then, for each edge, embed the edge in a unique half-plane with boundary the line. For this type of embedding to have meaning, then, we must impose some restrictions.

Before we discuss these restrictions we briefly mention regular projections as a way to represent embeddings in three dimensions. A regular projection of an embedding is a projection of the embedding in a plane in which there are only a finite number of points in which the projections of arcs cross and at each such point exactly two arcs cross. We also present three transformations, the *Reidemeister moves* in Figure 1.6. Two regular projections represent the same embedding if and only if they are related by a sequence of Reidemeister moves [34].

One natural restriction is to require that any two disjoint cycles have linking



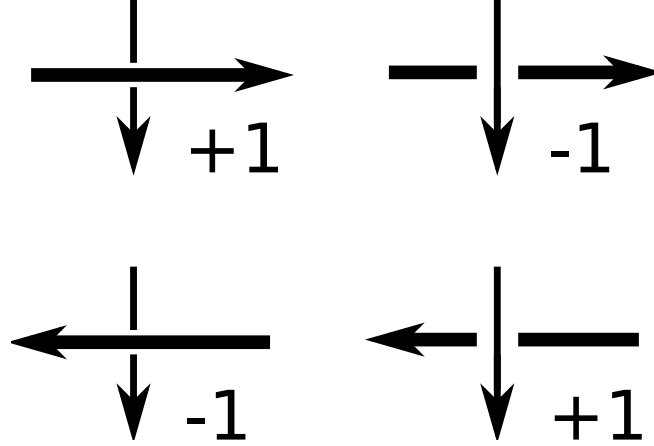


Figure 1.7: Linking Numbers

number zero. The *linking number* of two vertex-disjoint cycles is computed by taking a regular projection of the two cycles, orienting them arbitrarily and then, for each intersection, adding either  $+1$  or  $-1$  to the linking number as appropriate according to Figure 1.7. The linking number is then one half of this sum.

An embedding of a graph in which every two disjoint cycles satisfy this property is called *linkless*. These embeddings arose in work by Conway and Gordon [12] who proved that the complete graph  $K_6$  has no linkless embedding. A related notion is that of a *flat embedding*: one in which every cycle bounds a disk disjoint from the rest of the graph. Every flat embedding is then a linkless embedding, but the converse is false. A form of the converse is true, however, every graph that admits a linkless embedding also admits a flat embedding. This follows from a structure theorem of Robertson, Seymour, and Thomas:

**Theorem 1.4.3.** *For a graph  $G$  the following conditions are equivalent.*

- (1)  *$G$  has a flat embedding*
- (2)  *$G$  has a linkless embedding*
- (3)  *$G$  has no minor isomorphic to a member of the Petersen family*

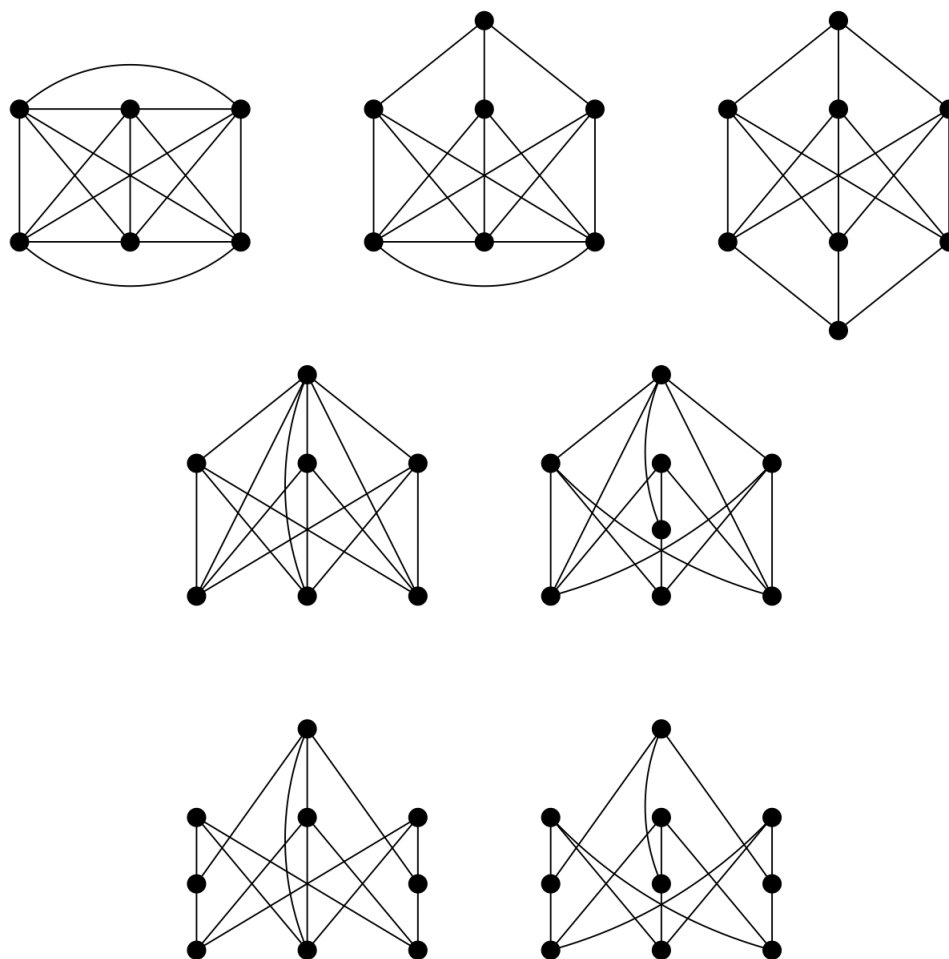


Figure 1.8: The Petersen Family

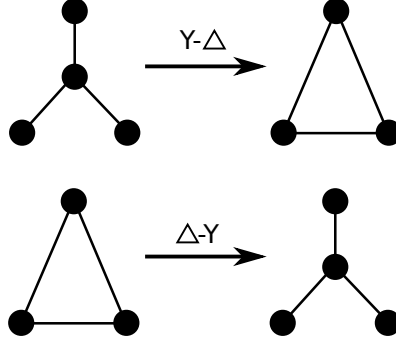


Figure 1.9:  $Y - \Delta$  and  $\Delta - Y$  transformations

Here, the Petersen family is the set of seven graphs pictured in Figure 1.8. These are the seven graphs obtainable from  $K_6$  by means of a sequence of  $Y - \Delta$  or  $\Delta - Y$  transformations. A  $Y - \Delta$  transformation replaces a degree three vertex with a triangle and a  $\Delta - Y$  transformation is its inverse. These operations are depicted in Figure 1.9. We note that both of these transformations preserve the property of being linkless (or flat).

In many regards, flat embeddings are more convenient than linkless ones and are a natural 3-dimensional analogue for planar embeddings. There are several natural examples of flat graphs. First, planar graphs are naturally flat: a planar embedding embedded in  $\mathbb{S}^2$  is flat. Another example is the class of *apex* graphs, those graphs  $G$  that contain a vertex  $v$  such that  $G \setminus \{v\}$  is planar. In this case, we embed  $G \setminus \{v\}$  planarly, and then embed  $v$  off of that plane. For each edge of  $v$ , we embed it as the natural straight-line between  $v$  and the other end of the edge. Then this embedding is flat (and hence linkless). There are, however, graphs that are far from being planar that are also linkless. We formalize that by saying that a graph  $G$  is *k-almost planar* if  $G$  contains a set of  $k$  vertices  $X$  such that  $G \setminus X$  is planar. In Chapter 6, we exhibit a class of graphs such that for any fixed  $k$  an infinite number of them are not  $k$ -almost planar.

On the algorithmic side, Theorem 1.4.3 and the Graph Minor Containment Algorithm of [37] provide an  $O(n^3)$  algorithm to test whether a particular graph has a flat or linkless embedding. These theorems, however, do not seem to provide a way to construct such an embedding in polynomial time. In the case of linkless embeddings, such a polynomial-time algorithm has been found by van der Holst [50] using an algebraic approach. In the case of flat embeddings, however, a polynomial time algorithm is claimed in [24], but the description of the algorithm misses a major ingredient. In that paper, the authors reduce the problem to 4-connected graphs. Such graphs have at most one flat embedding up to a homeomorphism of  $\mathbb{S}^3$ . Then they invoke [40, Theorem 1.2], which implies that an embedding of one Kuratowski subgraph (a subdivision of  $K_5$  or  $K_{3,3}$ ) determines the embedding of every other Kuratowski subgraph, and they conclude that this gives a way to embed the entire graph uniquely. While the embedding is, in fact, unique, in order to obtain an algorithm we require output that is the presentation of this embedding. Finding and presenting such an embedding is far from trivial and seems unlikely from the information they present.

## 1.5 A Summary of Results

For the convenience of the reader, we summarize here the main results of this thesis, along with some of the necessary definitions.

The main theorem of Chapter 2, proven as Theorem 2.1.4 the following:

**Theorem 1.5.1.** *Every planar graph without cycles of length four through six or eared seven cycles is 3-colorable.*

Here, we say that a cycle,  $C$ , is *eared* if there exist vertices  $a, b, c, d$  in order around  $C$  such that the edges  $ab$  and  $cd$  are in triangles. In fact we show the following strengthening:

**Theorem 1.5.2.** *Let  $G$  be a connected plane graph without cycles of length four through six or eared seven cycles and with an outer cycle of length at most 11. Suppose no interior vertex is adjacent to three vertices of the outer cycle. Then any proper 3-coloring of the subgraph induced by the outer cycle extends to a proper 3-coloring of  $G$ .*

The primary purpose of Chapter 3 is to prove the following result of interest both independently and for its use in Chapter 4. Here, we say that a subdivision is *odd* if each path that represents a subdivided edge has odd length. We say that a bipartite graph  $G$  is *internally 4-connected* if it is 3-connected, has at least five vertices, and there is no partition  $(A, B, C)$  of  $V(G)$  such that  $|A|, |B| \geq 2$ ,  $|C| = 3$  and  $G$  has no edge with one end in  $A$  and the other in  $B$ . Then we have that

**Theorem 1.5.3.** *Every internally 4-connected bipartite non-planar graph contains an odd subdivision of  $K_{3,3}$ .*

Odd subdivisions of  $K_{3,3}$  are of particular importance in Chapter 4. Here we provide a simpler proof of a theorem of McCuaig, Robertson, Seymour, and Thomas [31, 42]:

**Theorem 1.5.4.** *A brace has a Pfaffian orientation if and only if it is isomorphic to the Heawood graph, or if it can be obtained from planar braces by repeated application of the trisum operation.*

On the way, we prove the following three theorems from [42] in a way that leads to a polynomial time algorithm to decide if a bipartite graph is Pfaffian:

**Theorem 1.5.5.** *Let  $G$  be a nonplanar brace. Then  $G$  contains one of  $K_{3,3}$ , the Heawood graph, or Rotunda as a matching minor.*

**Theorem 1.5.6.** *Let  $G$  be a brace not isomorphic to the Heawood graph that contains the Heawood graph as a matching minor. Then  $G$  contains  $K_{3,3}$  as a matching minor.*

**Theorem 1.5.7.** *Let  $G$  be a brace that contains Rotunda as a matching minor. Then either  $G$  contains a set  $X$  of four vertices such that  $G \setminus X$  has three components or  $G$  contains  $K_{3,3}$  as a matching minor.*

In Chapter 5, we consider flat embeddings, specifically with an eye towards a polynomial-time algorithm to find a flat embedding. We show the following result:

**Theorem 1.5.8.** *There exists an absolute constant  $N$  such that if  $G$  is a graph on at least  $N$  vertices at least one of the following holds:*

- (1)  *$G$  contains a graph in the Petersen family as a minor*
- (2)  *$G$  contains a complete separation*
- (3)  *$G$  contains a peripheral theta, the deletion of whose arc leaves  $G$  Kuratowski-connected*
- (4) *There exists  $X \subseteq V(G)$ ,  $|X| \leq 1$  such that  $G \setminus X$  is planar*

We discuss each of these outcomes briefly. The first means that  $G$  does not have a flat embedding by Theorem 1.4.3. The fourth gives a natural flat embedding by saying that  $G$  is either apex or planar.

The second requires several definitions. First, we say that a graph  $H$  is a  $Y - \Delta$  minor of  $G$  if it can be formed from  $G$  by a sequence of contractions, deletions, or  $Y - \Delta$  operations. Let  $(A, B)$  be a separation in a graph  $G$  with  $K = A \cap B$ ,  $K^*$  complete on  $K$ . Then the separation is complete if  $|K| \leq 4$ ,  $G \setminus K$  has exactly two components if  $|K| = 4$ , and  $G$  contains  $A \cup K^*$  and  $B \cup K^*$  as  $Y - \Delta$  minors. Such complete separations are useful because, by a theorem of [41], a flat embedding of  $G$  can be formed naturally from flat embeddings of  $A \cup K^*$  and  $B \cup K^*$ .

For the third outcome, we again need several definitions. First, we say that a graph is a theta graph if it is isomorphic to a cycle plus one additional edge between two vertices not adjacent in the cycle. This additional edge is the *arc* of the theta

graph. We say that  $G$  contains a peripheral theta if  $G$  contains an induced subgraph  $C$  isomorphic to a theta graph such that  $G \setminus C$  is connected. Finally, a graph is *Kuratowski-connected* if it is 3-connected and, for every separation  $(A, B)$  with  $|A \cap B| \leq 3$  there is a planar embedding of either  $G[A]$  or  $G[B]$  with  $A \cap B$  embedded on the outer face. A *panel* for a cycle in a flat embedding is a disc with boundary that cycle disjoint from the rest of the graph. This result is of interest since we show

**Theorem 1.5.9.** *Let  $G$  be a flatly embeddable graph. Assume that  $G$  has a peripheral theta graph with cycle  $C$  and arc  $e$ , and assume also that  $G \setminus e$  is Kuratowski connected. Let  $\phi$  be a flat embedding of  $G \setminus e$ , let  $\Delta$  be a panel for  $C$  in the embedding  $\phi$ , and let  $\psi$  be an embedding of  $G$  such that  $\psi(e) \subseteq \Delta$  and  $\psi(x) = \phi(x)$  for every vertex and edge  $x$  of  $G \setminus e$ . Then  $\psi$  is a flat embedding of  $G$ .*

This means that there is a natural way to take a flat embedding for a  $G \setminus e$  where  $e$  is the arc of a peripheral theta and use it to construct a flat embedding for  $G$ .

In Chapter 6, we consider the question of whether there exist flat graphs which are far from planar. Specifically, we are interested in whether there exists a family of graphs that are well-connected such that for any  $k$  there are infinitely many graphs in the family that are not planar after deleting any  $k$  vertices. We answer this question in the affirmative for 5 connected graphs and present such a family. We note that by [26] and [25], sufficiently large 6-connected flat graphs are either planar or apex, so this is, in a sense, best possible.

## CHAPTER II

### STEINBERG'S CONJECTURE

#### 2.1 *Introduction*

This chapter is concerned with a partial result in the direction of Steinberg's Conjecture. Proposed in 1976, Steinberg's Conjecture [46] deals with three colorability of planar graphs:

**Conjecture 2.1.1.** (Steinberg) Let  $G$  be a planar graph that contains no four or five cycles. Then  $G$  is 3-colorable.

As discussed in more detail in the introduction, this conjecture is tight in the sense that we need to exclude both four cycles and five cycles. There have been several different techniques over the past 40 years in developing partial results in the direction of this conjecture, though they tend to fall into two distinct categories. The first, in a vein proposed by Paul Erdős is to exclude cycles of length 4 through  $k$  for some values of  $k$ . This culminated in the following theorem (though it was later improved to only exclude cycles of length 5, and 7 and triangles that share an edge [9]):

**Theorem 2.1.2.** (*Borodin et al*) [10] *Every planar graph without cycles of length from 4 to 7 is 3-colorable.*

The other approach is to exclude some subset of these cycles and to look at the minimum distance between two triangles. If we only exclude five cycles, Borodin et al [11] showed in 2003 that any planar graph without five cycles and without triangles at a distance less than four is three colorable, a result later strengthened by Xu to only exclude triangles at a distance less than 3 [52]. A recent paper by Borodin and Glebov [8] improves this further to



**Theorem 2.1.3.** *(Borodin and Glebov) [8] Every planar graph without cycles of length 5 and with the minimum distance between triangles at least 2 is three colorable.*

Our result is in a similar vein but requires a preliminary definition.

**Definition.** Let  $C$  be a cycle and  $a, b, c, d$  consecutive vertices in order around  $C$ . Then we say that  $C$  is *eared* if the edges  $ab$  and  $cd$  are both in triangles and we refer to those triangles as *ears*.

**Theorem 2.1.4.** *Every planar graph without cycles of length four through six or eared seven cycles is 3-colorable.*

Note that this gives a theorem that is strictly stronger than Theorem 2.1.2, though still incomparable with Theorem 2.1.3. In proving this statement, we show several interesting new reductions that may have application to further results in settling Steinberg's Conjecture.

## 2.2 The Theorem

In order to prove Theorem 2.1.4, we prove the following stronger theorem which immediately implies it.

**Theorem 2.2.1.** *Let  $G$  be a connected plane graph without cycles of length four through six or eared seven cycles and with an outer cycle of length at most 11. Suppose no interior vertex is adjacent to three vertices of the outer cycle. Then any proper 3-coloring of the subgraph induced by the outer cycle extends to a proper 3-coloring of  $G$ .*

**Definition.** The graph  $H^*$  is the graph on 12 vertices formed by taking the 11-cycle  $v_0, v_1, v_2, \dots, v_{10}$  and adding an additional vertex  $v$  adjacent to  $v_0, v_1$ , and  $v_6$ .

**Definition.** Let  $G$  be a plane graph and  $D$  the boundary cycle of its outer face. Then we say that a vertex  $v$  of  $G$  is *interior* if  $v$  is not on  $D$ .

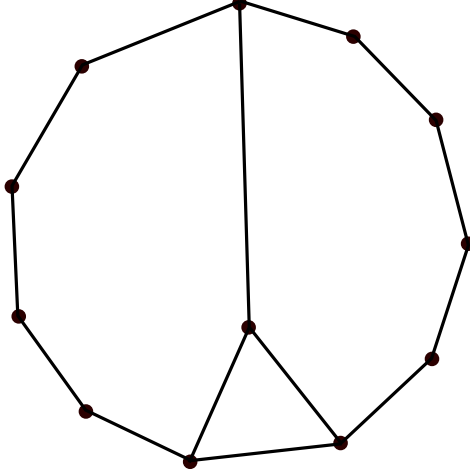


Figure 2.1: The graph  $H^*$

Assume that  $G$  is a counterexample with the minimum number of vertices and that  $G$  has the minimum number of edges among the counterexamples with  $|V(G)|$  vertices. Let  $D = d_1d_2, \dots, d_m, m \leq 11$  be the boundary cycle of the outer face  $f_0$  of  $G$ . We can now immediately see several structural properties of  $G$ .

(P1) Every internal vertex of  $G$  has degree at least 3.

(P2)  $G$  is 2-connected.

(P3)  $G$  has no separating cycle of length at most 10.

(P4) Every separating cycle of  $G$  of length 11 has one side of the separation that is exactly one vertex with three neighbors on the cycle.

(P5)  $D$  has no chord.

### 2.3 *Reducible Preliminaries*

Our strategy at this point is as follows: we show a number of structures that cannot exist in a minimal counterexample and then use a discharging argument based on Euler's Formula to show that one of them must appear. In this section we provide several lemmas and small reducible configurations that will make handling the long list of larger reducible configurations possible. The following two lemmas and their corollary will be used to show that a reduction is valid; that is, after making a

particular identification, the resulting graph satisfies the conditions of the theorem so we may apply induction.

**Lemma 2.3.1.** *Let  $G$  be a plane graph satisfying the conditions of the theorem as well as  $(P3)$  and  $(P4)$ . Let  $v_0, v_1, v_2, v_3, v_4$  be the vertices in a path  $P$  in order with  $v_1, v_2, v_3$  interior vertices and not both  $v_0$  and  $v_4$  on  $D$ . Let  $U$  be a set of vertices containing  $v_1, v_2$ , and  $v_3$  and not  $v_0$  or  $v_4$ . Further, suppose there is another path,  $Q$  between  $v_0$  and  $v_4$  disjoint from  $P$  and of length  $\geq 4$  such that not all neighbors of  $v_1, v_2, v_3$  are contained in the cycle formed by the union of these two paths, and that any path between  $v_0$  and  $v_4$  of length at most 6 and internally disjoint from  $P$  either forms a separating cycle when completed by  $P$  or contains a vertex of  $U$ . Then either the graph formed by identifying  $v_0$  and  $v_4$  and deleting the vertices of  $U$  satisfies the conditions of the theorem or there is a path,  $R$  in  $G$  of length 7 between  $v_0$  and  $v_4$  such that after identifying  $v_0$  and  $v_4$   $R$  is an eared 7-cycle.*

*Proof.* Let  $H$  be the graph formed after identifying  $v_0$  and  $v_4$ . Then  $H$  must violate the conditions of the theorem. If  $H$  has a cycle of length at most 6, that must come from a path of length at most 6 between  $v_0$  and  $v_4$ , but such a path gives a separating cycle of length at most 10 when joined with  $P$ , so cannot exist.

Let  $d_0$  be the distance between  $v_0$  and the outer cycle and  $d_1$  the distance between  $v_4$  and the outer cycle. Let  $x_0$  and  $x_1$  be the nearest neighbors in the outer cycle to  $v_0$  and  $v_4$  respectively.

If  $d_0 + d_1 \leq 1$ , then the cycle formed by taking  $P$  with  $x_0, x_1$  and the path between them in the outer cycle is separating, so has length at least 11, so the corresponding path in the outer cycle has length at least 6. Since this is true on either side, the total length of the outer cycle is at least 12 which is impossible.

Suppose after identification we have a subgraph isomorphic to  $H^*$ . Then without loss of generality, either  $v_0$  has a neighbor with two edges to the outer cycle or  $v_0$  itself does. Suppose  $v_0$  has two neighbors in the outer cycle,  $c_0$  and  $c_1$ . Then consider

the cycle formed by the path  $c_0v_0Pv_4x_1$  and completed by the outer cycle. This is separating, so has length at least 11, so the path in the outer cycle has length at least 5. On the other side, the cycle formed by  $c_1v_0Pv_4x_1$  and the outer cycle separates at least two vertices of  $Q$ , so must have length at least 12, so the path in the outer cycle has length at least 6. Since the path between  $c_0$  and  $c_1$  has length at least 1, the outer cycle has length at least 12 which is a contradiction.

□

**Lemma 2.3.2.** *Let  $G$  be a graph satisfying the conditions of the theorem as well as properties three and four above. Let  $v_0, v_1, v_2, v_3, v_4$  be consecutive vertices in a path  $P$  with  $v_1, v_2, v_3$  all interior and not both  $v_0$  and  $v_4$  on  $D$ . Suppose  $v_0$  and  $v_1$  are both adjacent to the same vertex  $x$  and  $v_2$  and  $v_3$  are both adjacent to the same vertex  $y$ . Suppose that  $xv_1v_2v_3v_4$  is not part of the boundary of a face of length at most 10. Finally suppose that if  $Q$  is a path between  $v_0$  and  $v_4$  of length at least 7 that does not pass through  $v_1, v_2, v_3$ , or  $x$ , then the cycle  $v_0Qv_4Pv_0$  separates  $x$  and  $y$  in  $G$ . Then the graph formed by identifying  $v_0$  and  $v_4$  and deleting  $v_1, v_2, v_3$  satisfies the conditions of Theorem 2.2.1.*

*Proof.* Let  $H$  be the graph formed by the identification. We show first that  $H$  has no new cycle of length at most 7. Such a cycle must correspond to a path of length at most 7 in  $G$  between  $v_0$  and  $v_4$ . It must then separate  $x$  and  $y$  from one another unless it passes through  $x$ . If it passes through  $x$ , it either separates some other vertex from  $y$  or forms the boundary of a face with  $xv_1v_2v_3v_4$  of length at most 10 which is impossible by assumption. If it passes through  $x$  and separates some other vertex, this gives a separating cycle of length at most 10 which violates property 3. Otherwise, it gives a separating cycle of length at most 11 which, by property 4 must be isomorphic to  $H^*$ . But then the face of  $H^*$  containing either  $v_0v_1v_2y$  or  $xv_1v_2v_3v_4$  would be an eared 7-cycle in  $G$  which is forbidden.

We show next that the total distance between  $v_0$  and  $v_4$  and the outer cycle is at

least 3. Suppose  $v_0$  is on the outer cycle and  $v_4$  is at distance at most 2 from  $t$ , a vertex on the outer cycle. Then the length of the outer cycle between  $v_0$  and  $t$  on the side with  $y$  must be at least 6 (since we cannot have an 11 cycle that separates  $y$  as shown in the previous paragraph). Similarly, the length of the outer cycle between  $v_0$  and  $t$  on the side with  $x$  must be at least 6 since it either passes through  $x$  and so must give a cycle of length at least 11 or separates  $x$  and so must give a cycle of length at least 12. In either case, the length of the outer cycle is then at least 12. The analysis is identical for  $v_4$  on the outer cycle and  $v_0$  at distance at most 2 from the outer cycle and for each a distance 1 from the outer cycle.

We show next that the coloring of the outer cycle is still proper. In order for the coloring to no longer be proper, we must have at least one of  $v_0$  and  $v_4$  on the outer cycle and the other at distance at most 1 away from it which is impossible.

Finally, we need to show that no vertex is now adjacent to three vertices of the outer cycle. If the vertex in  $H$  adjacent to three vertices of the outer cycle is not the new vertex, then it must be a neighbor of either  $v_0$  or  $v_4$  and the other must be on the outer cycle, so one is distance 2 from the outer cycle and the other is on it which is impossible. If the new vertex is adjacent to three vertices of the outer cycle, then  $v_0$  and  $v_4$  must both be adjacent to the outer cycle which is impossible.

So  $H$  satisfies the conditions of the theorem.  $\square$

**Corollary 2.3.3.** *Let  $G$  be a graph satisfying the conditions of the theorem as well as properties three and four above. Let  $v_0, v_1, v_2, v_3, v_4$  be internal vertices, consecutive in a path  $P$ . Suppose  $v_0$  and  $v_1$  are both adjacent to the same vertex  $x$  and  $v_2$  and  $v_3$  are both adjacent to the same vertex  $y$  with  $x$  and  $y$  internal vertices. Finally suppose that if  $Q$  is a path between  $v_0$  and  $v_4$  that does not pass through  $v_1, v_2, v_3$ , or  $x$ , then the cycle  $v_0Qv_4Pv_0$  separates  $x$  and  $y$  in  $G$ . Then the graph formed by identifying  $v_0$  and  $v_4$  and deleting  $v_1, v_2, v_3, x$  satisfies the conditions of Theorem 2.2.1.*

*Proof.* We first delete all the edges incident with  $x$  except those going to  $v_1$  and  $v_0$ .

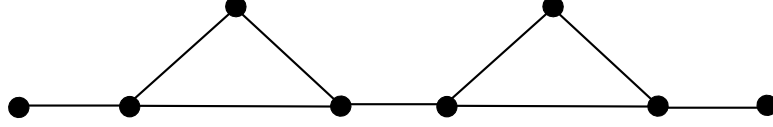


Figure 2.2: A tetrad

This new graph  $H$  satisfies all the same properties of the theorem and has no new separating cycles. Further, if  $xv_1v_2v_3v_4$  is part of the boundary in  $H$  of a face of length at most 10, that boundary must actually contain  $v_0xv_1v_2v_3v_4$ , so  $v_0v_1v_2v_3v_4$  along with this face gives a separating cycle of length 9 in  $G$ . By Lemma 2.3.2 applied to  $H$ , deleting  $v_1v_2v_3$  and identifying gives a graph satisfying the properties of the theorem. We can then delete  $x$  and continue to get a graph satisfying the conditions of the theorem.  $\square$

**Lemma 2.3.4.** *Let  $v_0, v_1, v_2, v_3$  be consecutive vertices along a path in  $G$ . Further, let  $x$  be adjacent to  $v_1$  and  $v_2$  and let both  $v_1$  and  $v_2$  have degree three. Then any coloring in which  $v_0$  and  $v_3$  have different colors extends to  $v_1$  and  $v_2$ .*

*Proof.* We may assume  $v_0$  is colored 1 and  $v_3$  is colored 2. If  $x$  is colored 1, color  $v_2$  and then  $v_1$ . If  $x$  is colored 2, color  $v_1$  and then  $v_2$ . If  $x$  is colored 3, color  $v_1$  2 and  $v_2$  1.  $\square$

We make use of the following definition from [10]:

**Definition.** Let  $v_0, v_1, \dots, v_5$  be consecutive vertices along the boundary of a face in which  $v_1, v_2, v_3, v_4$  all have degree 3. Let  $x$  be adjacent to both  $v_1$  and  $v_2$  and  $y$  be adjacent to  $v_3$  and  $v_4$ . Then we say that  $v_0, \dots, v_5, x, y$  forms a *tetrad*.

**(P6)**  $G$  has no tetrad.

*Proof.* Suppose that  $v_0, \dots, v_5, x, y$  is a tetrad. Then identify  $x$  and  $v_5$  and delete  $v_1, \dots, v_4$ . Then by Corollary 2.3.3 and induction the resulting graph is three colorable. So we need to show that the coloring extends to  $G$ . Color  $v_1$ , then  $v_2$ . Then the color

on  $v_2$  is different from that of  $x$ , so is different from that of  $v_5$ , so we can color  $v_1$  and  $v_4$  by Lemma 2.3.4.  $\square$

We now consider a number of larger subgraphs that must be excluded from  $G$ . For the sake of notation and simplicity we present the following definitions:

**Definition.** A *configuration*  $(G, f, b)$  consists of a graph  $G$  along with a map  $f : V(G) \rightarrow \mathbb{Z}$  and a map  $b : V(G) \rightarrow \{0, 1\}$  with  $f(v) \geq \deg(v)$ .

We say that a graph  $G$  *includes* a configuration  $(H, f, b)$  if  $G$  contains a subgraph  $G'$  isomorphic to  $H$  with isomorphism  $\sigma$  such that if  $v \in V(G')$  with  $b(\sigma(v)) = 0$  then  $\deg(v) = f(\sigma(v))$  and if  $b(\sigma(v)) = 1$  then  $\deg(v) \geq f(\sigma(v))$ . A graph *contains* a configuration  $(H, f, b)$  if it includes that configuration and the corresponding isomorphism is the identity. We represent configurations by drawings of graphs where a vertex  $v$  is filled in if  $b(v) = 0$  and open if  $b(v) = 1$ . We represent  $f(v)$  by the number of edges or half-edges incident with  $v$  in the drawing.

**(P7)** If  $G$  contains the configuration corresponding to the diagram on the left of Figure 2.3 with  $v_0, v_1, \dots, v_6$  interior vertices, then it contains either the configuration in the center of Figure 2.3, the configuration on the right of Figure 2.3, or the configuration in the center of Figure 2.3 with  $u_3$  of degree at least 3 and one of  $t_2, u_1, u_2, u_3$  on the outer face.

*Proof.* We identify  $v_0$  with  $t_2$  and delete  $v_4, v_5, v_6$ . By Lemma 2.3.1, the resulting graph satisfies the conditions of the theorem unless there is a path of length 7 between  $v_0$  and  $t_2$  with triangles that become a distance 2 apart after identification.

If the resulting graph satisfies the conditions, we can color it. Then color  $v_4$ , then  $v_5$  and  $v_6$  by Lemma 2.3.4. So we must have a path of length 7 between  $v_0$  and  $t_2$  with triangles that become distance 2 apart after the identification. Suppose the path is  $t_2, u_1, u_2, u_3, u_4, u_5, u_6, v_0$ . Then  $u_3$  is adjacent to  $t_1$ . So the triangles cannot be on

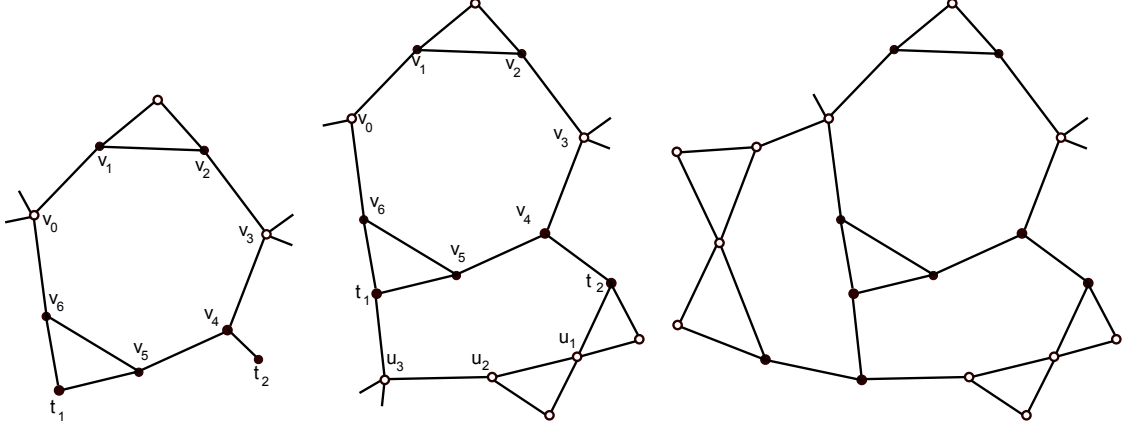


Figure 2.3: Configurations for Property 7

the edge  $u_2u_3$  or  $u_3u_4$ . We will show later (see Configuration 2 below) that  $v_0$  is not in a triangle, so we cannot have a triangle on  $v_0u_6$ . So the only remaining option is on  $u_5u_6$  and  $u_1t_2$ . Note that  $t_1$  must be degree 3.

We now identify  $u_3$  with  $v_3$  and delete  $v_6v_5v_4$ , and  $t_1$ . Such a graph satisfies the conditions of the theorem unless there is a path of length 7 between  $v_3$  and  $u_3$  with triangles distance 2 apart after identification. If we can color the resulting graph, color  $v_4$ , then  $v_6$ , then  $v_5$  and  $t_1$  by Lemma 2.3.4. So we must have the path of length 7 and triangles distance 2 apart. Similar analysis to the above shows that one of the triangles must be on the edge  $u_1u_2$ .

If  $u_3$  has degree at least 4, then this is exactly the middle configuration. Otherwise, we identify  $u_2$  with  $v_0$  and delete  $u_3, t_1, t_2, v_4, v_5, v_6$ . By assumption  $u_1$  and  $t_2$  are not on the outer cycle, so by Lemma 2.3.1 the resulting graph satisfies the conditions of the theorem unless there is a path of length 7 with close triangles after identification. But the only way this can happen is exactly the configuration on the right of Figure 2.3. So the resulting graph must be three colorable. Then color  $u_3, t_2, v_4, v_5$  and then  $t_1$  and  $v_6$  by Lemma 2.3.4.

□

We include a list of 16 configuration diagrams which cannot be contained in  $G$ .



Note that configurations 7 and 8 can be found in [10] as reducible configurations in the context of their similar theorem. In the next section we will handle each of these individually, but they are included together here for reference.

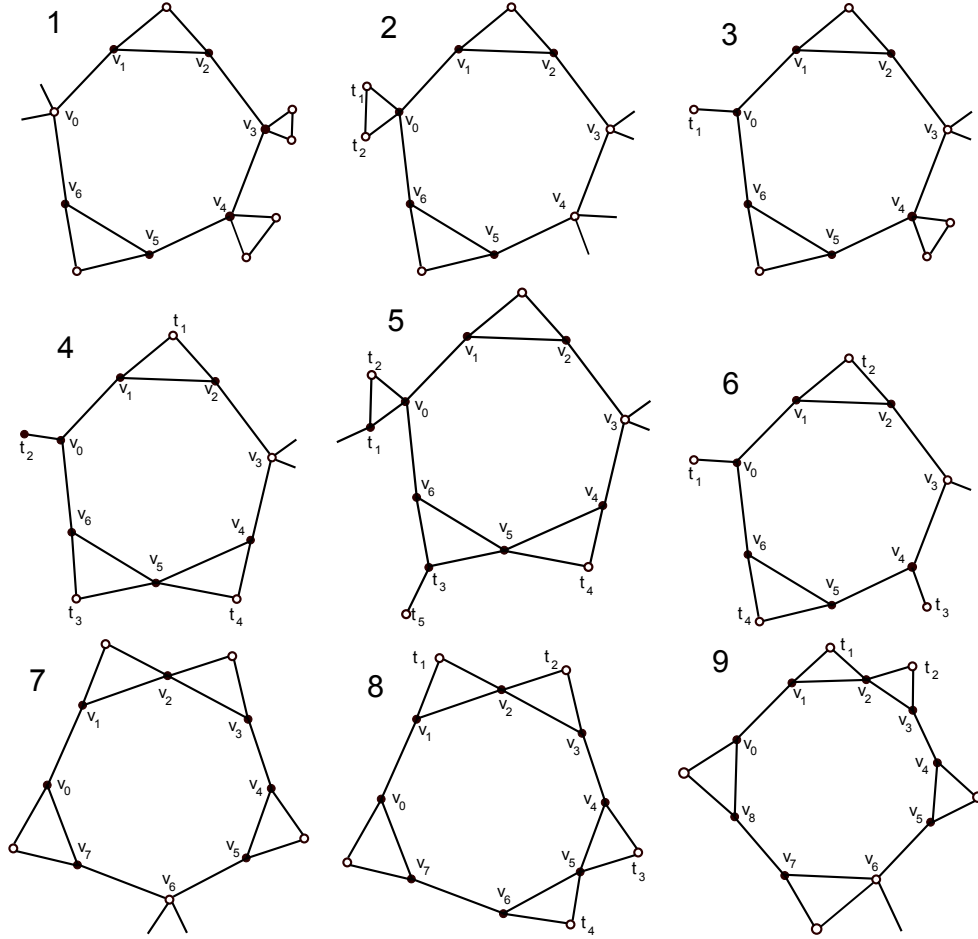


Figure 2.4: Configurations 1 through 9

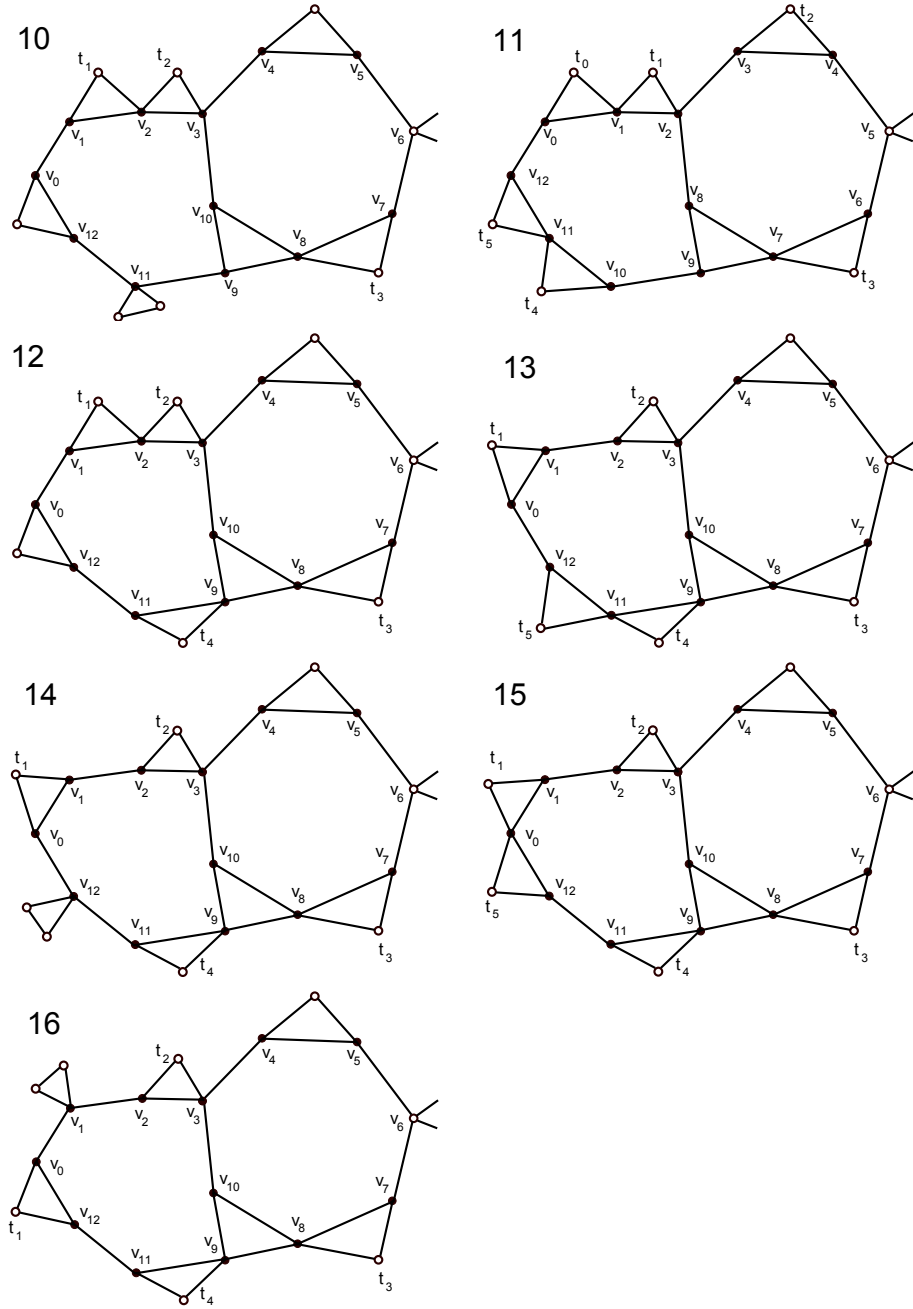


Figure 2.5: Configurations 10 through 16

## 2.4 Reducing the Reducible Configurations

We now describe the reductions for the configurations presented in Figures 2.4 and 2.5.

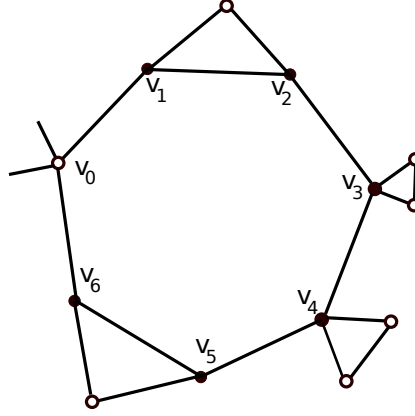


Figure 2.6: Configuration 1

### Configuration 1

$G$  does not contain the configuration in Figure 2.6 with  $v_0$  through  $v_6$  interior vertices.

*Proof.* Suppose otherwise. Then delete  $v_1, v_2, v_5$ , and  $v_6$  and add edges  $v_0v_3$  and  $v_0v_4$ . We argue that this graph satisfies the properties of the theorem so that we may apply induction.

First, if there is now a cycle of length 4 through 6, it must come from a path between  $v_0$  and  $v_3$  (or equivalently  $v_0$  and  $v_4$ ) of length at most 5, which would give a separating 8 cycle in the original graph. A new 7 cycle would similarly give a separating 9 cycle. We do create a new triangle, but the only incident cycles that might be of length 7 would need to use the path  $t_2v_4v_3t_1$ , so if they are of length 7 are eared 7-cycles in  $G$ .

Second, we cannot have created a chord in the outer cycle since the only edges we added were between interior vertices, and similarly we cannot have made any interior vertex adjacent to three vertices of the outer cycle.

So by induction, we can now color the graph except for  $v_1, v_2, v_5$ , and  $v_6$ . But  $v_0$  and  $v_4$  are different colors, so by Lemma 2.3.4 we can color  $v_5$  and  $v_6$  and similarly since  $v_0$  and  $v_3$  are different colors we can color  $v_1$  and  $v_2$ .  $\square$

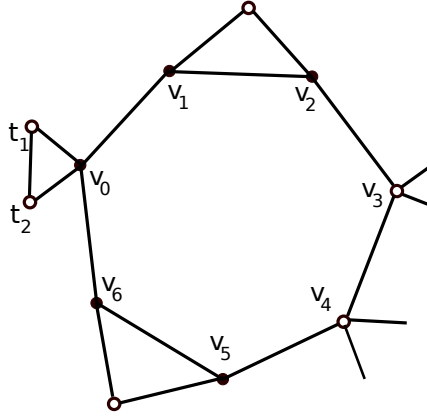


Figure 2.7: Configuration 2

### Configuration 2

$G$  does not contain the configuration in Figure 2.7 with  $v_0$  through  $v_6$  interior vertices.

*Proof.* We delete  $v_0, v_1, v_2, v_5$ , and  $v_6$  and identify  $t_1$  with  $v_3$  and  $t_2$  with  $v_4$ . We note that either identification still gives a graph that satisfies the conditions of the theorem by Lemma 2.3.2. So the only possible problems must be a triangle on  $v_3v_4$ , but that cannot happen since it must come from a triangle on  $t_1t_2$  which would give a four cycle.

So we can color the graph except for  $v_0, v_1, v_2, v_5$ , and  $v_6$ . Color  $v_0$ . Then  $v_0$  is colored differently from  $v_3$  and  $v_4$ , so by Lemma 2.3.4 we can color  $v_1$  and  $v_2$  as well as  $v_5$  and  $v_6$ .  $\square$

### Configuration 3

$G$  does not contain the configuration in Figure 2.8 with  $v_0$  through  $v_6$  interior vertices.

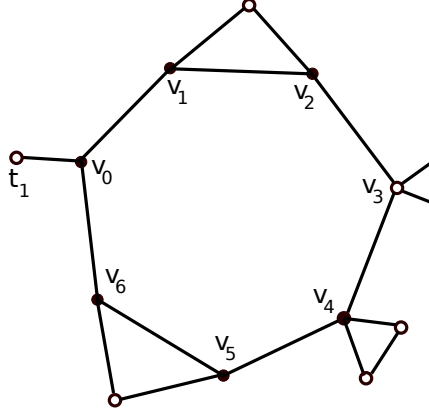


Figure 2.8: Configuration 3

*Proof.* We delete  $v_0, v_1, v_2, v_5$ , and  $v_6$  and identify  $t_1$  and  $v_4$ . We check that the resulting graph satisfies the properties of the theorem.

First, we cannot have created a short cycle, since any new cycle of length at most seven comes from a  $t_1$ - $v_4$  path of length at most 7 which gives a separating at most 11 cycle. If it is a separating 11 cycle, it must be  $H^*$  which is impossible since then the seven cycle incident with  $v_4$  and  $v_5$  has ears. Then by Lemma 2.3.1, we can color the resulting graph.

Then we color in order  $v_2, v_1, v_0$  and not that  $v_0$  is colored differently from  $v_4$ , so by Lemma 2.3.4 we can color  $v_5$  and  $v_6$ .  $\square$

#### Configuration 4

$G$  does not contain the configuration in Figure 2.9 with  $v_0$  through  $v_6$  interior vertices.

*Proof.* We delete  $v_0, v_1, v_2$ , and  $v_6$  and identify  $t_1$  and  $v_5$ . We check that the resulting graph satisfies the properties of the theorem.

First, we cannot have created a short cycle, since any new cycle of length at most seven comes from a  $t_1$ - $v_5$  path of length at most 7 which gives either a  $t_1$ - $t_3$  or  $t_1$ - $t_4$  path of length at most 6 which in turn gives a separating at most 10 cycle. hen by Lemma 2.3.1, we can color the resulting graph.

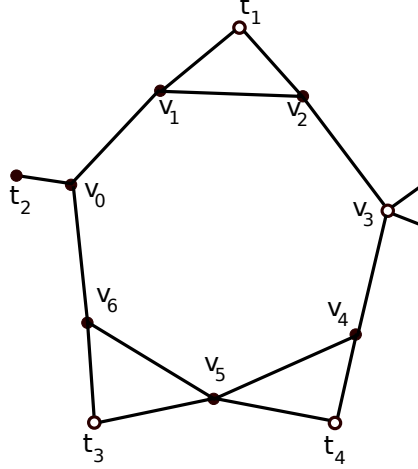


Figure 2.9: Configuration 4

We color  $v_6$  then  $v_0$ . If the color of  $v_0$  is not the same as the color of  $v_3$ , then the graph is colorable by Lemma 2.3.4. Further, if  $v_0$  or  $v_3$  is colored the same as  $t_1$ , then we can easily color  $v_1$  and  $v_2$ . So neither is colored the same as  $t_1$ .

So without loss of generality,  $t_1$  and  $v_5$  are both colored 1 and  $v_3$  is colored 2. We know that  $v_0$  must be colored 2 and  $v_6$  cannot be colored 1, so  $v_6$  must be forced to be 3 and  $t_2$  must be colored 1. Then  $t_3$  is colored 2. Finally,  $v_4$  must be colored 3 and  $t_4$  must be colored 2.

But then we change the coloring on  $v_4, v_5, v_6$ , and  $v_0$ . Specifically, color  $v_4$  with 1,  $v_5$  with 3,  $v_6$  with 1, then color  $v_2, v_1$ , and finally  $v_0$  in order.

□

### Configuration 5

$G$  does not contain the configuration in Figure 2.10 with  $v_0, \dots, v_6, t_1$  interior vertices.

*Proof.* We delete  $v_0, v_1, v_2, v_4, v_5, v_6, t_1$ , and  $t_3$  and identify  $t_2, t_5$ , and  $v_3$  together. Note that  $t_3$  is an interior vertex since otherwise 2 of its neighbors would be on the outer cycle. Any single identification of this set would give a graph that satisfies the conditions of the theorem by Lemma 2.3.2 or Lemma 2.3.1 except that there could

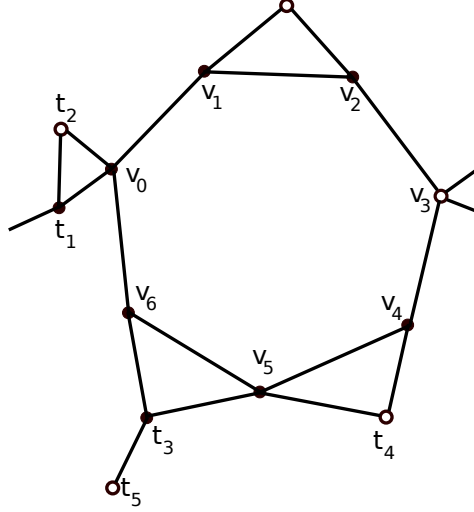


Figure 2.10: Configuration 5

be a path of length 7 with triangles that become a distance two apart between  $t_5$  and  $v_3$ .

If there is such a path of length 7, that gives a separating 11 cycle which must be  $H^*$ . So we have a path  $u_1, u_2, u_3, u_4, u_5, u_6$  with an edge  $t_4 u_3$ . For the graph after identification to not satisfy the conditions of the theorem, we must have two triangles on this path that become a distance 2 apart after identification. The edge  $t_5 u_1$  cannot have a triangle since that would give a pair of ears with the triangle on  $t_3 v_5$ . Similarly, the edges  $u_2 u_3$ ,  $u_3 u_4$ , and  $v_3 u_6$  cannot have triangles since they would give ears with the triangle  $t_4 v_5 v_4$ . So the only edges that could have triangles are  $u_1 u_2$ ,  $u_6 u_5$ , and  $u_4 u_5$ . But triangles on these edges cannot form ears after identification, so the resulting graph does not have any objectionable short cycles.

We then color the resulting graph by induction. Color  $t_2, v_3, t_5$  all 1, and without loss of generality, we can color  $t_1$  2. If  $t_4$  is colored 1, we can color  $v_4$  2,  $v_5$  3,  $t_3$  2,  $v_6$  1, and  $v_0$  3. Then we color  $v_1$  and  $v_2$  by Lemma 2.3.4.

If  $t_4$  is colored 2, we color  $v_4$  3,  $v_5$  1 and then can color  $v_0, v_6, t_3$  and then  $v_1$  and  $v_2$ .

If  $t_4$  is colored 3, we color  $v_4$  2,  $v_5$  1 and then can color  $v_0, v_6, t_3$  and then  $v_1$  and

$v_2$ .

□

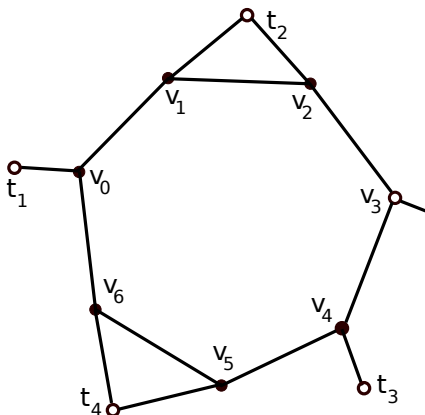


Figure 2.11: Configuration 6

### Configuration 6

$G$  does not contain the configuration in Figure 2.11 with  $v_0, \dots, v_6$  interior vertices.

*Proof.* We identify  $t_1$  with  $v_3$  and delete  $v_0, v_1, v_2, v_4, v_5, v_6$ . By Lemma 2.3.1, the resulting graph satisfies the conditions of the theorem unless there is a path of length 7 between  $t_1$  and  $v_3$ .

If the resulting graph satisfies the conditions, we can color it. Then color in order  $v_4, v_5, v_6, v_0$ , then  $v_1$  and  $v_2$  by Lemma 2.3.4. So we must have a path of length 7 between  $t_1$  and  $v_3$  which gives a separating 11-cycle. Suppose the path is  $t_1, u_1, u_2, u_3, u_4, u_5, u_6, v_3$ . Then  $u_3$  is adjacent to  $t_2$ . Note that  $t_2$  must have degree 3.

Suppose  $v_3$  has degree 3. Then delete  $v_0, v_1, v_2, v_3, v_4, v_5, v_6, t_2$  and color the remaining graph. We may assume  $t_4$  is colored 1. If we can color  $v_0$  1, then we color  $t_2$ , then  $v_1, v_2, v_3, v_4, v_5, v_6$  in order. So we cannot color  $v_0$  1, so  $t_1$  is colored 1. Similarly, if we could color  $v_4$  1, we then color  $v_3, v_2, t_2, v_1, v_0, v_6, v_5$  in order. So  $t_3$  is also 1. If the neighbor of  $v_3$  is not 1, then we color  $v_3$  1, then color  $v_4$  2 and  $v_0$  3. Then we can color  $t_2$ , then  $v_5, v_6$  and  $v_1, v_2$  by Lemma 2.3.4. So the neighbor of  $v_3$  is also 1. Then



color  $t_2$  a color other than 1, without loss of generality, we can say 2. Then color  $v_0$  and  $v_3$  2 and  $v_4$  3. Then  $v_1$  and  $v_2$  are colored 1 and 3, and we can color  $v_6$  3 and  $v_5$  2.

So  $v_3$  must have degree at least 4. Then identify  $u_3$  and  $v_4$ , and delete  $t_2, v_0, v_1, v_2, v_5, v_6$ . Either the resulting graph is three colorable or there is a path of length 7 between  $u_3$  and  $v_4$ . But such a path must create an 11 cycle that separates both the neighbors of  $v_3$  from  $v_1$  and  $v_0$ , so cannot exist by Property 3. So we can color the resulting graph. We may assume  $u_3$  and  $v_4$  are colored 1 and  $v_3$  is colored 2.

If  $t_1$  is not colored 3, then color  $v_0$  3, color  $t_2$ , then color  $v_1, v_2$  and  $v_5, v_6$  by Lemma 2.3.4. So  $t_1$  is colored 3. So color  $v_0$  2,  $t_2$  2, then  $v_1$  1,  $v_2$  3, then color  $v_5, v_6$  by Lemma 2.3.4.

□

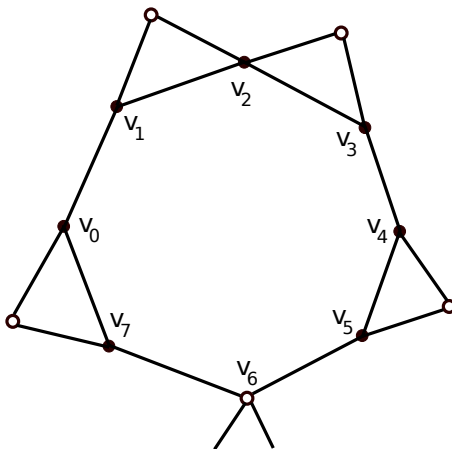


Figure 2.12: Configuration 7

### Configuration 7

$G$  does not contain the configuration in Figure 2.12 with  $v_0, \dots, v_7$  interior vertices.

*Proof.* Delete  $v_0, v_1, v_3, v_4, v_5$ , and  $v_7$  and identify  $v_2$  and  $v_6$ . Then the resulting graph satisfies the conditions of the theorem by Lemma 2.3.2, so is three colorable. Then

color  $v_3$  and  $v_1$  (which must be different colors than  $v_6$ , so by Lemma 2.3.4, we can color  $v_0, v_7$  and  $v_4, v_5$ .  $\square$

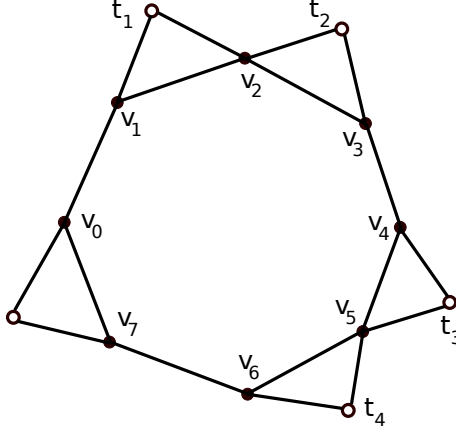


Figure 2.13: Configuration 8

### Configuration 8

$G$  does not contain the configuration in Figure 2.13 with  $v_0, \dots, v_7$  interior vertices.

*Proof.* Delete  $v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7$ , and  $v_8$  and identify  $t_1$  and  $t_3$ . Then by Lemma 2.3.2, the resulting graph is three colorable. We may assume that  $t_1$  and  $t_3$  are both colored 1.

Suppose  $t_2$  is colored 1. If  $t_4$  is colored 1 as well, color  $v_6$  3, then color  $v_5, v_4, v_3, v_2, v_1$ . Then  $v_1$  is colored 2, so by Lemma 2.3.4 we color  $v_0$  and  $v_7$ . If  $t_4$  is colored 2, then we color  $v_6$  1,  $v_5$  3, and  $v_4$  2. Then color  $v_3, v_2, v_1$ . Since  $v_1$  is not colored 1 we can then color  $v_0$  and  $v_7$ .

Suppose  $t_2$  is colored 2. Then  $v_1$  is colored 2,  $v_2$  3, and  $v_3$  1. If  $t_4$  is colored 1, then by symmetry this is one of the previous cases. Otherwise, we can color  $v_5$ , then  $v_4$  and  $v_6$ . Then  $v_6$  is colored 1, so we can color  $v_0$  and  $v_7$ .

$\square$

### Configuration 9

$G$  does not contain the configuration in Figure 2.14 with  $v_0, \dots, v_8$  interior vertices.

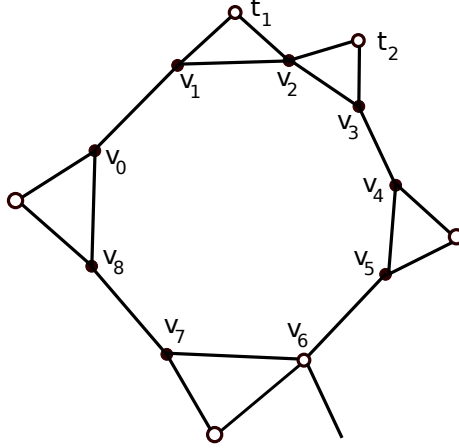


Figure 2.14: Configuration 9

*Proof.* Identify  $t_2$  and  $v_6$  and delete  $v_0, v_1, v_2, v_3, v_4, v_5, v_7$ , and  $v_8$ . By Lemma 2.3.2 and induction, the resulting graph is three colorable. We now extend the coloring to  $G$ .

Color  $t_2$  and  $v_6$  both 1. Then color  $v_7$ . Without loss of generality, we assume  $v_7$  is colored 2. If  $t_1$  is colored 1, color  $v_1$  3, then color  $v_2, v_3$ , then  $v_4, v_5$  and  $v_0, v_8$ . If  $t_1$  is colored 2 or 3, then color  $v_2, v_1, v_3, v_4, v_5, v_0, v_8$  in order, noting that the color on  $v_1$  is always 1 and that the color on  $v_3$  is never 1.  $\square$

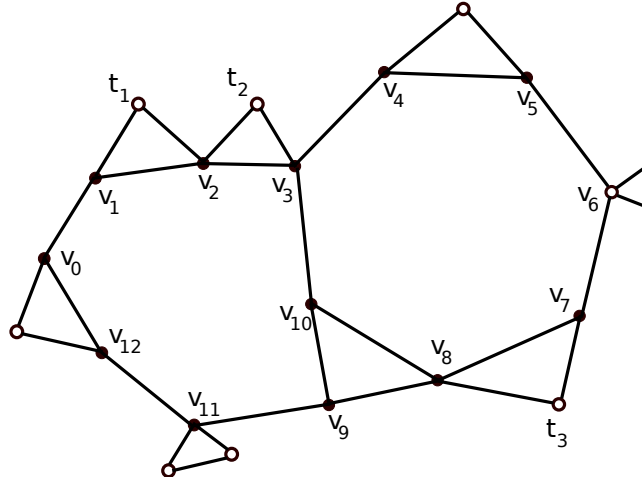


Figure 2.15: Configuration 10

**Configuration 10**

$G$  does not contain either the configuration in Figure 2.15 with  $v_0, \dots, v_{12}$  interior vertices or the configuration in Figure 2.15 with  $v_0, \dots, v_{12}$  interior vertices and  $v_{11}$  degree 3.

*Proof.* Identify  $v_2, v_6$ , and  $v_{11}$  and delete all the other  $v_i$ . We note first that there cannot be paths of length 7 between  $v_{11}$  and  $v_6$  with triangles a distance two apart after identification. If there were such a path and  $v_{11}$  is in a triangle, then the resulting  $H^*$  has an eared seven cycle. If  $v_{11}$  is degree 3, then there is nowhere on the path for the violating triangles to go without creating ears. So by Lemma 2.3.1, the resulting graph satisfies the properties of the theorem, so is three colorable.

We now show that a coloring extends. Color  $v_2, v_{11}$ , and  $v_6$  all 1. Then color  $v_1$  and  $v_3$ , then color  $v_7$  and  $v_8$ . Then  $v_3$  is not colored 1, but  $v_{11}$  is, so we can color  $v_{10}$  and  $v_9$ . Similarly,  $v_6$  is colored 1 but  $v_3$  is not, so we can color  $v_4$  and  $v_5$ . And finally,  $v_1$  is not colored 1, so we can color  $v_0$  and  $v_{12}$ .  $\square$

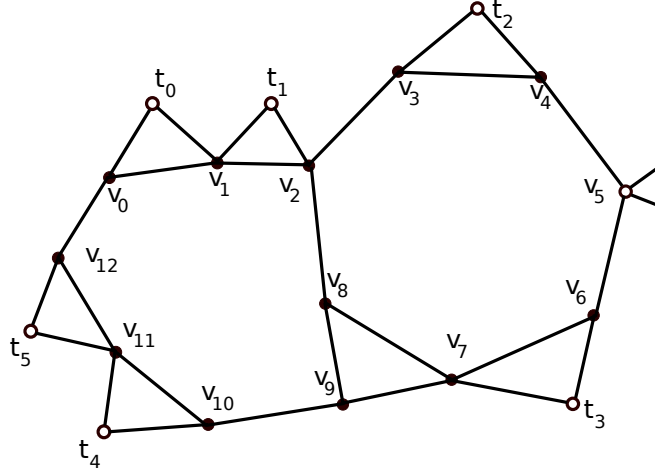


Figure 2.16: Configuration 11

### Configuration 11

$G$  does not contain the configuration in Figure 2.16 with  $v_0, \dots, v_{12}$  interior vertices.

*Proof.* Identify  $t_1$  with  $v_5$  and  $t_0$  with  $t_4$ , then delete all the other  $v_i$ . Then by Lemma 2.3.2, the resulting graph satisfies the properties of the theorem, so is three colorable.

We now show that a coloring extends. Without loss of generality, color  $t_1$  and  $v_5$  1. Suppose  $t_0$  and  $t_4$  are colored 1. If  $t_5$  is colored 1 as well, color  $v_{10}$  2, then color  $v_{11}, v_{12}, v_0, v_1, v_2$  (then  $v_2$  is colored 3. Then color  $v_6, v_7$ , so we can color  $v_8, v_9$  and  $v_3, v_4$  by Lemma 2.3.4. If  $t_5$  is colored 2, color  $v_2$  3,  $v_1$  2,  $v_0$  3, then color  $v_{11}$  3,  $v_{12}$  1,  $v_{10}$  2. Then we can again color  $v_6, v_7$  then  $v_8, v_9$  and  $v_3, v_4$ .

So we have  $t_0$  and  $t_4$  colored 2. Then  $v_0$  is colored 1 and  $v_2$  is colored 2. If  $t_5$  is colored 1, we color  $v_{11}$  3,  $v_{12}$  2, and  $v_{10}$  1. If  $t_5$  is colored 2, we color  $v_{11}$  1,  $v_{12}$  3, and  $v_{10}$  3, and if  $t_5$  is colored 3, we color  $v_{11}$  1,  $v_{12}$  2, and  $v_{10}$  3. In any case,  $v_{10}$  and  $v_2$  have different colors, so we can color  $v_6$  then  $v_7$ , then  $v_8, v_9$ , and  $v_3, v_4$ .

□

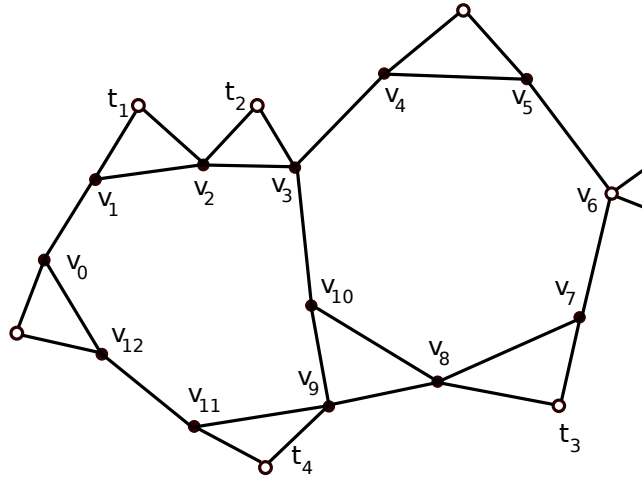


Figure 2.17: Configuration 12

### Configuration 12

$G$  does not contain the configuration in Figure 2.17 with  $v_0, \dots, v_{12}$  interior vertices.

*Proof.* Identify  $t_2, t_4$ , and  $v_6$ , then delete all the other  $v_i$ . Then by Lemma 2.3.2, the resulting graph satisfies the properties of the theorem, so is three colorable.

We now show that a coloring extends. Without loss of generality,  $t_2$ ,  $t_4$ , and  $v_6$  are colored 1. Suppose  $t_1$  is colored 1. If  $t_3$  is not colored 1, then color  $v_7$ , then color  $v_8$  (which must be colored 1. Then we can color  $v_{11}$  2 and  $v_1$  3. Then color  $v_2$  2,  $v_3$  3,  $v_9$  3, and then  $v_{10}$  2. Then we can color  $v_4$  and  $v_5$  as well as  $v_0$  and  $v_{12}$ .

If  $t_3$  is colored 1, then color  $v_{11}$  2 and then color  $v_7, v_8, v_9$ . Then  $v_{10}$  will be colored 1. Color  $v_1$  3, then color  $v_2, v_3$ , then  $v_4, v_5, v_0, v_{12}$ .

So we may assume  $t_1$  is not colored 1, so 2. Then color  $v_2, v_3, v_1$  (so  $v_1$  is 1 and  $v_3$  is 2). Then color  $v_7, v_8$ . If  $v_8$  is a 1, then color  $v_{10}, v_9, v_{11}$ , and then  $v_0, v_{12}, v_4, v_5$ . If  $v_8$  is not a 1, then color  $v_9$  and note that none of  $v_3, v_8, v_9$  are 1, so we can color  $v_{10}$  1. Then color  $v_9$ , then  $v_0, v_{12}, v_4, v_5$ .

□

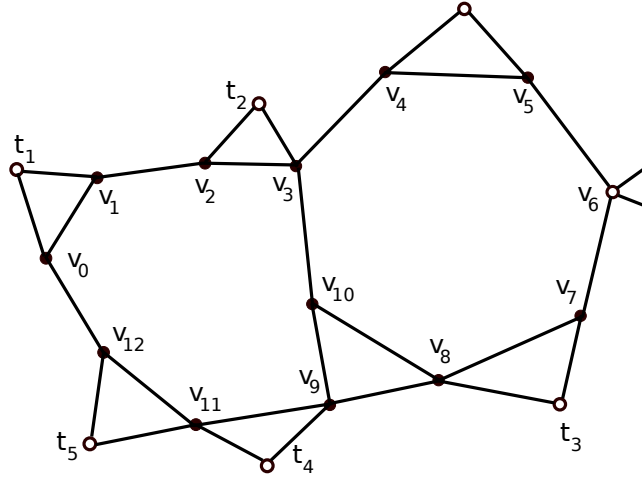


Figure 2.18: Configuration 13

### Configuration 13

$G$  does not contain the configuration in Figure 2.18 with  $v_0, \dots, v_{12}$  interior vertices.

*Proof.* Identify  $t_2$ ,  $t_4$ , and  $v_6$ , then delete all the other  $v_i$ . Then by Lemma 2.3.2, the resulting graph satisfies the properties of the theorem, so is three colorable.

We now show that a coloring extends. Without loss of generality,  $t_2$ ,  $t_4$ , and  $v_6$  are colored 1. Suppose  $t_5$  is colored 1 and  $t_3$  is colored 1. Then color  $v_{12}$  2, and color

$v_{11}, v_9, v_8, v_7$ . Then color  $v_{10}$  1,  $v_2$  3, and then color  $v_3, v_4, v_5, v_0, v_1$ . So suppose  $t_3$  is not colored 1. Then color  $v_7$  and  $v_8$  will be colored 1. Then color  $v_2$  2, and color  $v_3, v_{10}, v_9, v_{11}, v_{12}$ . Then  $v_{12}$  will be colored 3, so we can color  $v_0, v_1, v_4, v_5$ .

So we may assume  $t_5$  is colored 2. Then we color  $v_{11}, v_{12}, v_9$  with  $v_{12}$  as 1 and  $v_9$  as 2. We then have  $v_6$  and  $v_9$  different colors, so we can color  $v_7$  and  $v_8$ . Then color  $v_{10}, v_3, v_2$ , then  $v_4, v_5, v_0, v_1$ .

□

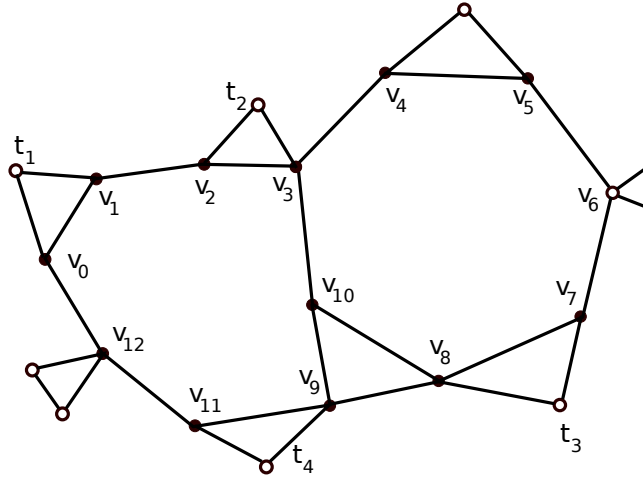


Figure 2.19: Configuration 14

### Configuration 14

$G$  does not contain either the configuration in Figure 2.19 with  $v_0, \dots, v_{12}$  interior vertices or the configuration in Figure 2.19 with  $v_0, \dots, v_{12}$  interior vertices and  $v_{12}$  degree 3.

*Proof.* Identify  $t_2, t_4$ , and  $v_6$ , then delete all the other  $v_i$ . Then by Lemma 2.3.2, the resulting graph satisfies the properties of the theorem, so is three colorable.

We now show that a coloring extends. Without loss of generality,  $t_2, t_4$ , and  $v_6$  are colored 1. Color  $v_{12}$ . If  $v_{12}$  is colored 1, then color  $v_{11}, v_9, v_7$  and  $v_8$  (by Lemma 2.3.4), then  $v_{10}, v_3, v_2$ , then we can color  $v_0, v_1, v_4, v_5$ . So we may assume  $v_{12}$  is colored

2. Then color  $v_{11}, v_9$  (so  $v_9$  is colored 2). If  $t_3$  is colored 1, color  $v_7, v_8, v_{10}$  (then  $v_{10}$  is colored 1). Then color  $v_2$  3 and  $v_3$  2, then color  $v_0, v_1, v_4, v_5$ .

So we may assume  $t_3$  is not colored 1. Then color  $v_7$  and then we can color  $v_8$  with 1. Then color  $v_{10}, v_3, v_2$  and  $v_2$  will be colored 3, so we can color  $v_0, v_1, v_4, v_5$ .

□

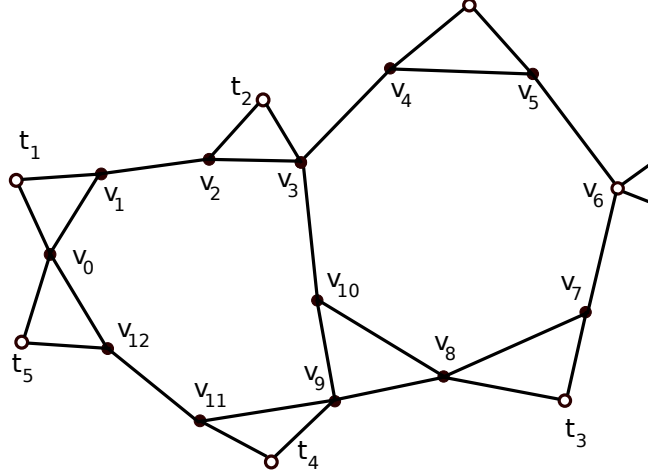


Figure 2.20: Configuration 15

### Configuration 15

$G$  does not contain the configuration in Figure 2.20 with  $v_0, \dots, v_{12}$  interior vertices

*Proof.* Identify  $t_1$  with  $t_4$  and then  $t_2$  with  $v_6$  then delete all the other  $v_i$ . Then by Lemma 2.3.2, the resulting graph satisfies the properties of the theorem, so is three colorable.

We now show that a coloring extends. Without loss of generality,  $t_2$ , and  $v_6$  are colored 1. Suppose first that  $t_1$  and  $t_4$  are also colored 1. Then if  $t_3$  is not colored 1, color  $v_7, v_8$  which will be 1. If  $t_5$  is also 1, color  $v_1$  with 3, then  $v_0, v_{12}, v_{11}, v_9, v_{10}$  which will be 2, so we can color  $v_2, v_3, v_4, v_5$ . If  $t_5$  is not 1, so 2, then color  $v_0$  3,  $v_1$  2, and  $v_{12}$  1. Then color  $v_{11}$  3,  $v_9$  2,  $v_{10}$  3, so  $v_2, v_3, v_4, v_5$  can be colored.

If  $t_3$  is colored 1, we color  $v_0, v_1, v_{12}, v_{11}, v_9, v_8, v_7, v_{10}$ . Note that  $v_{10}$  is colored 1, so since  $v_1$  is not colored 1, color  $v_2, v_3$ , then  $v_4, v_5$ .



So we may assume  $t_1$  and  $t_4$  are not colored 1, so are colored 2. If  $t_5$  is a 1, then  $v_0$  is 3,  $v_1$  is 1,  $v_{12}$  is 2. So we can take  $v_{11}$  is 1,  $v_9$  is 3, so we can color  $v_7, v_8, v_{10}, v_3, v_2$ , then  $v_4, v_5$ . If  $t_5$  is a 2, we color  $v_1$  3,  $v_0$  1,  $v_{12}$  3, then  $v_{11}$  is colored 1,  $v_9$  3, then we can color  $v_7, v_8$  and  $v_{10}$  is not colored 3, so is different from  $v_1$ . Then we can color  $v_2, v_3$ , so  $v_4, v_5$ .

So we may assume  $t_5$  is colored 3. Then  $v_1$  is 3,  $v_0$  is 1,  $v_{12}$  is 2 then we can color  $v_{11}$  1,  $v_9$  3, then we can color  $v_7, v_8$  and  $v_{10}$  is not colored 3, so is different from  $v_1$ . Then we can color  $v_2, v_3$ , so  $v_4, v_5$ .

□

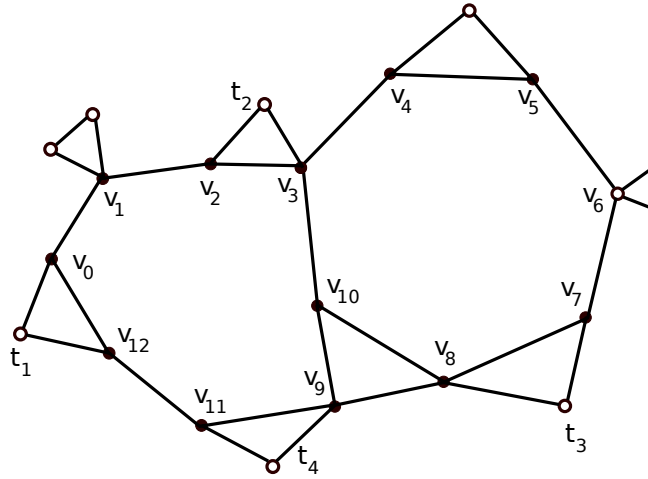


Figure 2.21: Configuration 16

### Configuration 16

$G$  does not contain either the configuration in Figure 2.21 with  $v_0, \dots, v_{12}$  interior vertices or the configuration in Figure 2.15 with  $v_0, \dots, v_{12}$  interior vertices and  $v_1$  degree 3.

*Proof.* Identify  $t_2, t_4$ , and  $v_6$ , then delete all the other  $v_i$ . Then by Lemma 2.3.2, the resulting graph satisfies the properties of the theorem, so is three colorable.

We now show that a coloring extends. Without loss of generality,  $t_2, t_4$ , and  $v_6$

are colored 1. Color  $v_1$ . If  $v_1$  is colored 1. then just color in order  $v_7, v_8, v_9, v_{11}, v_{10}, v_3, v_2, v_4, v_5, v_0, v_{12}$ . So  $v_1$  is colored 2. Then color  $v_{11}$  with 3,  $v_9$  with 2, then we can color  $v_7$  and  $v_8$ , so we can color  $v_{10}$  something other than 2 (so not the same as  $v_1$ ). So we can color  $v_2, v_3$ , and so  $v_4, v_5$ . Since  $v_1$  and  $v_{11}$  are not the same color, we can then color  $v_0, v_{12}$ .  $\square$

## 2.5 Discharging

To complete the proof we show that there is no planar graph with all the properties of the minimal counterexample from the previous section. To do so, we consider an immediate consequence of Euler's Formula:

$$\sum_{v \in V(G)} (\deg(v) - 4) + \sum_{f \in F(G)} (|f| - 4) = -8 \quad (5)$$

To each vertex we assign a charge equal to its degree minus four, to each interior face we give a charge equal to the length of its bounding cycle minus four, and to the outer cycle we give a charge equal to its length plus four. Then the total charge in the graph is exactly zero.

**Definition.** An interior vertex is said to be *bad* if it has degree three and is incident with a triangle.

We now provide a set of rules in the same vein as [10] to move the charge around after which each interior face and vertex will have nonnegative charge and the outer face will have strictly positive charge.

(R1) Each vertex sends  $1/3$  charge to each incident face of size three.

(R2)

a) Each non-triangular interior face sends  $2/3$  charge to each incident interior vertex of degree three in a triangle and to each incident vertex of degree 2.

b) Each non-triangular interior face sends  $1/3$  charge to each incident interior vertex of degree three not in a triangle.

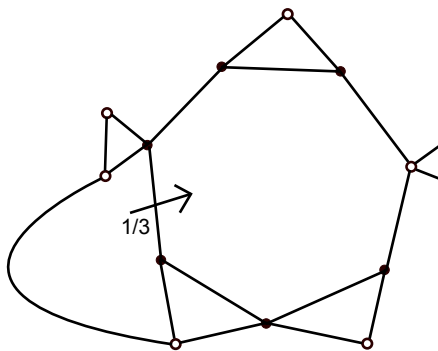


Figure 2.22: Rule 5

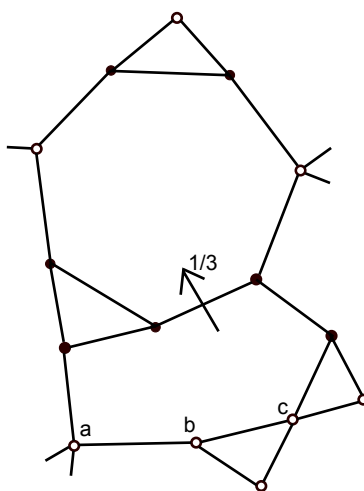


Figure 2.23: Rule 6a

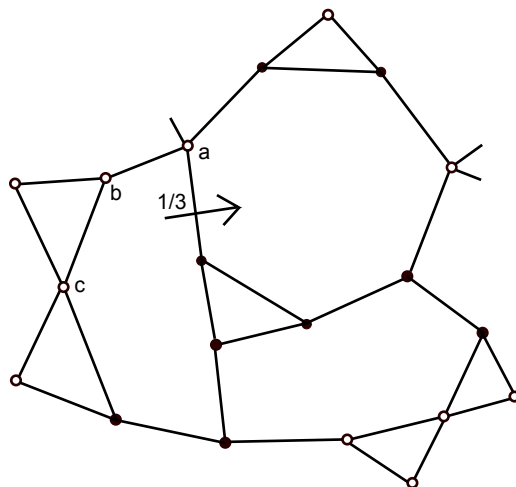


Figure 2.24: Rule 6b

c) Each non-triangular interior face sends  $1/3$  charge to each incident interior vertex of degree four in two triangles adjacent to the same face.

d) Each non-triangular interior face sends  $1/3$  charge to each incident interior vertex of degree four in a triangle not adjacent to the face.

**(R3)**

a) Each non-triangular interior face takes  $1/3$  charge from each incident vertex of degree five adjacent to two adjacent triangles or to one triangle not adjacent to the face.

b) Each non-triangular interior face takes  $1/3$  charge from each incident vertex of degree at least six.

c) Each non-triangular interior face takes  $1/3$  charge from each incident vertex of degree four on the outer face if it shares an edge with the outer face and  $2/3$  charge otherwise.

**(R4)** The outer face gives  $4/3$  charge to each incident vertex.

**(R5)** Each interior face in a structure as shown gives  $1/3$  charge as indicated in Figure 2.22.

**(R6)** Each interior face in a structure as shown gives  $1/3$  charge as indicated in Figures 2.23 and 2.24.

a) This face requires that either  $a$  have degree at least 4 or one of  $a$ ,  $b$ , or  $c$  be on the outer face.

b) Here vertex  $a$  has degree at least 4.

We now need to check that each interior face and vertex have nonnegative charge and that the outer face has positive charge. Since the total charge in the graph stays the same after the application of any of these rules and started at exactly zero, this would provide a contradiction and show that there no minimum counterexample can exist.

For the outer face, the charge is  $|f| + 4$  and it loses at most  $4/3|f|$ , so the total

charge is  $4 - 1/3|f|$ . Since  $f$  is at most 11, this is strictly positive.

**Lemma 2.5.1.** *After applying the discharging rules, the charge on each vertex is at least 0.*

*Proof.* Vertices of degree at least 6 start with charge  $\deg(v) - 4$  and give away at most  $1/3$  to each incident face, so at most  $\deg(v)/3$  and so are left with  $2/3 \deg(v) - 4$  charge which is at least 0.

Vertices of degree 5 start with charge 1. They can be incident with at most two triangles. If they meet no triangles, they end with 1 charge since none of the rules apply. If they meet one triangle, they give away  $1/3$  charge by rule 1 and  $2/3$  charge by rule 3 so are left with 0. If they meet two triangles, they give away  $2/3$  charge by rule 1 and  $1/3$  by rule 3, so are left with 0 charge.

Vertices of degree 4 start with no charge. Suppose first that they are interior. If they are incident with no triangles, none of the rules apply and they end with 0 charge. If incident with one triangle, they give away  $1/3$  by rule 1 but receive  $1/3$  by rule 2d so end at 0. If incident with two triangles, they give away  $2/3$  by rule 1 and then receive  $2/3$  by rule 2c so end with 0. If they are on the outer face, they receive  $4/3$  charge by rule 4, and give away at most  $4/3$  charge between rules 1 and 3c.

Vertices of degree 3 start with  $-1$  charge. Suppose first that they are interior. If incident with no triangles, they receive  $1/3$  from each incident face by rule 2b so end at 0 charge. If incident with one triangle, they give away  $1/3$  by rule 1 and take  $2/3$  from each other incident face by rule 2a and end at 0. If they are on the outer face, they receive  $4/3$  by rule 4 and give away at most  $1/3$  by rule 1.

Vertices of degree 2 start with  $-2$  charge. They can only be incident with the outer face, so take  $4/3$  by rule 4 and  $2/3$  by rule 2a and so end at 0.  $\square$

We now need to check the charge on each face. We begin with the internal faces.

We note first that only faces of size 7 lose charge to rule 6. Further, a face can

lose to rule 5 at most once for each degree 4 vertex it has. Note further that those degree 4 vertices do not take any other charge from the face, so degree four vertices take at most  $1/3$  charge.

So for faces of size at least 12, each vertex takes at most  $2/3$  charge, but the face starts with  $|f| - 4$  charge, so the total charge is  $1/3|f| - 4 \geq 0$ .

For faces of size 11, by parity not every vertex can be a degree three in a triangle, so at least one vertex only takes  $1/3$  charge, so the total charge is at least  $11 - 4 - 20/3 - 1/3 = 0$ .

For faces of size 10, we note that 5 consecutive degree three vertices in a triangle is a tetrad, so there can be at most 8 such vertices. So the total charge is at least  $10 - 4 - 16/3 - 2/3 = 0$ .

For faces of size 9, we can have at most 7 vertices of degree three in triangles without forming a tetrad. If both of the other vertices require  $1/3$  charge, the face would then be negative. So the degree three vertices in triangles must be split 4 and then 3, so we must have exactly Configuration 9.

For faces of size 8, we can have at most 6 bad vertices. We first look at faces that would be negative without rule 5. Suppose there are 5 bad vertices. Then each of the other vertices needs to take  $1/3$  charge or the face has nonnegative charge. Such vertices have either two or zero incident triangular edges, but each bad vertex is incident with exactly one triangular edge, so by parity this situation cannot exist. So there must be six bad vertices split either three and three or four and two.

Suppose they split four and two. Then the four vertices must form triangles with the two good vertices or else we have a tetrad. One of these good vertices must be incident with a second triangle or else the face has nonnegative charge. So the graph must be Configuration 8. If they split 3 and 3, we must again have one of the good vertices draw charge from the face, so it must be involved in two triangles. This gives Configuration 7.

We now consider the possibility of eight faces that give charge under rule 5.

Let the boundary of such a face,  $F$ , be  $v_0, v_1, a, b, c, d, e, f$ . Without loss of generality, we assume  $v_0$  is the degree 4 vertex that is shared with the seven cycle to which the face gives charge and  $v_1$  is the degree 3 vertex shared with the seven cycle. In this case, we say that  $v_1$  *supports* the instance of rule 5. Note that for this face to be negatively charged, there must be at least 5 bad vertices. Also, not both of  $a$  and  $f$  can be degree 3 by Configuration 5.

Suppose there are 6 bad vertices. Then either  $a$  or  $f$  is not bad and the rest of the vertices are bad. But then we have 5 vertices in a row that are all bad, so we have a tetrad which is impossible.

So we must have exactly 5 bad vertices. For such a face to be negatively charged, each other vertex must take  $1/3$  charge (viewing a rule 5 application as the degree 4 vertex that is shared taking  $1/3$  charge).

**Lemma 2.5.2.** *The face  $F$  gives at most once under rule 5*

*Proof.* Suppose first that vertex  $f$  supports a second instance of rule 5. Then  $e$  must be degree 3 and the edge  $ed$  is in a triangle. Then not all of  $a, b, c$ , and  $d$  are bad (since we have exactly five bad vertices). If  $d$  is not bad, then all of  $a, b, c$  must be bad which is impossible since if  $b$  and  $c$  are in a triangle together, then  $v_1, a, b, c$  is a tetrad. So we may assume  $d$  is bad. If  $c$  is not bad, it must support another instance of rule 5. But  $b, d, e$  are all bad, so this is an instance of Configuration 5. If  $b$  is not bad, then either it supports another instance of rule 5 which is Configuration 5 or  $a$  is not bad. Finally, if  $a$  is not bad, then not all of  $b, c, d$  can be bad. So this case is impossible, so  $f$  cannot support a second instance of rule 5

Next we suppose that  $a$  supports a second instance of rule 5. So then  $b$  is degree 3 and one of  $c, d, e, f$  is not bad. If any of  $c, d, e$  give under rule 5, they form Configuration 5 since the other two as well as  $b$  and  $f$  must be bad. If  $c$  is in two triangles,

all of  $d, e, f$  cannot be bad. Neither  $d$  nor  $e$  can be in two triangles if  $c$  and  $f$  are bad. And if  $f$  is in two triangles, then  $c, d, e$  cannot be all bad.

So either  $a$  or  $f$  is in two triangles and one of  $b, c, d$ , or  $e$  supports another rule 5 and the rest of the vertices are bad.

Suppose  $a$  is in two triangles. If  $b$  supports another rule 5, then it is impossible for  $c, d, e, f$  to all be bad. If  $c$  supports another rule 5 and  $b$  is bad, not both of  $e$  and  $f$  can be bad. If  $d$  supports the rule 5 and  $b$  and  $c$  are bad, then not both of  $e$  and  $f$  can be bad. Finally, if  $e$  supports the rule 5 and  $f$  is bad, then not all of  $b, c, d$  are bad.

So we must have  $f$  in two triangles and  $a$  bad. If both  $b$  and  $c$  are bad, we have a tetrad with  $v_1$  and  $a$ . If  $b$  supports the rule 5, then not all of  $c, d, e$  can be bad. If  $c$  supports the rule 5, then similarly not both of  $d, e$  can be bad. So it is impossible for this face to support multiple instances of rule 5 and still be negatively charged.

□

So we now have that  $f$  gives exactly once under rule 5. At least one of  $a$  and  $f$  is not bad. Suppose neither is bad. Then both are in triangles and the remaining vertices are bad. But this violates Configuration 12. So either  $a$  or  $f$  is bad.

Suppose  $a$  is bad and  $f$  is not bad. One of  $b, c, d, e$  is not bad. Note that if both  $b$  and  $c$  are bad, we have a tetrad  $(v_1, a, b, c)$ . So we may assume one of  $b$  or  $c$  is not bad. Suppose it is  $b$ . Then  $b$  must be degree 4 in one triangle or degree 3 and  $cd$  must be a triangular edge. But this gives Configuration 10. So  $c$  must not be bad. Then  $c$  must be in two triangles and  $b, d, e$  all bad. But this is Configuration 11.

So we may assume  $a$  is not bad and  $f$  is bad. If  $b$  is not bad, then it is in two triangles and  $de$  is a triangular edge. But this is Configuration 13. If  $c$  is not bad, then it is degree three or degree four in a triangle and  $de$  is a triangular edge. But this is Configuration 14. If  $d$  is not bad, it must be in two triangles, but this is Configuration 15. Finally, if  $e$  is not bad, it must be degree three or degree four in a



triangle and  $cd$  a triangular edge, but this is Configuration 16.

So there is no such 8-face.

For faces of size 7 we note first that rule 5 cannot apply.

We consider a 7-face that gives under rule 6a,  $F$ . Suppose first that all the vertices are internal. Then, in the notation of that rule,  $a$  has degree at least 4, so takes no charge. If  $b$  has degree 3 and  $c$  has degree 4, then this face is exactly Configuration 4. If  $b$  has degree three and  $c$  has degree at least 5, then the total charge is  $3 - 2/3 - 2/3 - 1/3 - 2/3 + 1/3 - 2/3 - 1/3$  (where the last  $1/3$  is from rule 6a) which is 0. If  $b$  has degree at least 4, the total charge is  $3 - 2/3 - 2/3 - 1/3 - 2/3 - 1/3 - 1/3 = 0$ .

So we may assume one of the vertices is on the outer cycle. If  $a$  is on the outer face and is degree 4, then either one of its neighbors is on the outer face or by rule 3c it gives  $2/3$  charge back to  $f$ . In the latter case, the total charge is  $3 - 2/3 - 2/3 - 1/3 - 2/3 - 1/3 - 2/3 + 2/3 - 1/3 = 0$ . In the former, it gives  $1/3$  charge back to  $f$  but one of its neighbors does not take  $2/3$  charge, so the total charge is then  $1/3$ .

So we may assume  $a$  has degree 3. Then one of  $a$ ,  $b$ , or  $c$  is on the outer face. If  $c$  is on the outer face, either it gives  $2/3$  charge back to  $f$  by rule 3c or one of its neighbors on  $f$  is also on the outer face. If the former, the total charge is  $3 - 2/3 - 2/3 - 1/3 - 2/3 + 2/3 - 2/3 - 1/3 - 1/3 = 0$ . If the latter,  $c$  only gives  $1/3$  charge back, but one of its neighbors no longer takes  $2/3$  charge, so the total charge is  $1/3$ . So we may assume that  $c$  is not on the outer face. If  $b$  is on the outer face with degree 3, then  $a$  must be as well, so the total charge is at least 0. If  $b$  has degree at least 4 and is on the outer face, then it either provides  $1/3$  charge if  $a$  is also on the outer face or provides  $2/3$  charge. In either case, the total charge remains nonnegative. Finally, if  $a$  is on the outer face, then one of its neighbors is as well, so the total charge is again at least 0. So the face has nonnegative charge.

We now consider a 7-face giving under rule 6b. Again to avoid Configuration 4, either  $b$  or  $c$  has degree greater than drawn, so the total facial charge is positive.

We now consider 7 cycles giving under neither rule. We note that there can be at most two bad vertices in a row without violating the conditions for seven cycles, so there are at most 4 bad vertices. Similarly, we can see that there cannot be just three bad vertices and four vertices that each take  $1/3$  charge by parity as in the case for eight faces. So there must be exactly 4 bad vertices. Since we cannot have four triangles incident with a seven face, we have either two or three.

If we have two triangles, let  $F$  be the face and its bounding cycle be  $a, v_0, v_1, b, c, v_2, v_3$  with the  $v_i$  in triangles.

Two of  $a, b$ , and  $c$  must each take  $1/3$  charge. Suppose first that  $a$  is in a triangle not incident with  $F$ . If it has degree at least 5, then it gives back  $1/3$  charge by rule 3a, so the face has nonnegative charge. Otherwise, it has degree exactly 4 which is Configuration 2. So  $a$  is not in a triangle.

If  $a$  has degree 3, then we may assume  $b$  also takes  $1/3$  charge. If  $b$  is degree 4 in a triangle, then this is Configuration 3. If  $b$  is degree 3, then this is Configuration 5. So  $a$  cannot take  $1/3$  charge.

Therefore,  $b$  and  $c$  both take  $1/3$  charge. If both are degree four vertices in triangles, this is Configuration 1. So we may assume  $c$  has degree 3. Then by Property 7 and rule 6 it receives  $1/3$  charge from a neighboring face. So  $F$  does not have negative charge.

Finally, we consider the case of three triangles. Let  $F$  be such a face and let the boundary cycle of  $F$  be  $a, v_0, v_1, v_2, b, v_3, v_4$  with  $t_1$  adjacent to  $v_0$  and  $v_1$ ,  $t_2$  adjacent to  $v_1$  and  $v_2$ , and  $t_3$  adjacent to  $v_3$  and  $v_4$ .

At least one of  $a$  or  $b$  must take charge from the face, so we may assume it is  $a$ . If  $a$  is degree 3, this is Configuration 4. So  $a$  is in a triangle. But then this face receives charge by rule 5, so is not, in fact, negatively charged. If both  $a$  and  $b$  are in triangles, then  $F$  receives charge twice by Rule 5, so again is not negatively charged.

Lastly, we need to check the charge of faces incident with the outer face. We

showed earlier that faces giving charge under rule 5 or rule 6 are nonnegative even if they are on the outer face, so we do not need to consider those faces.

We note that any face without degree 2 vertices either has two vertices that take nothing from the face since they are on the outer cycle or has a degree 4 vertex which then gives the face  $1/3$  charge. So none of these faces are negative.

Therefore we need only consider faces with vertices of degree 2. Note that having a vertex of degree 2 means that there are two vertices on the outer face incident with the cycle that do not take any charge. For a face of size at least 8, that means that it loses charge at most  $2/3(|F| - 2)$ , so has charge left at least  $|F|/3 - 8/3$ . So if  $|F| \geq 8$ , it cannot be negative. So we may assume  $F$  is a seven-face.

If there is one vertex of degree 2, at most 3 of the remaining vertices are bad (since otherwise we have ears), so the total charge is at least 0. If there are two vertices of degree 2, at most two of the remaining vertices can be bad, so the total charge is at least 0.

If there are 3 vertices of degree 2, the rest of the outer cycle along with this cycle gives a separating ten cycle, so this is impossible. If there are 4 vertices of degree 2, we must have exactly  $H^*$ . Five vertices of degree 2 gives a chord in the outer cycle (unless  $G$  is exactly this seven cycle which similarly isn't a problem).

So no face or vertex has negative charge and the outer face has strictly positive charge, so such a graph cannot exist.

## CHAPTER III

### ODD $K_{3,3}$ SUBDIVISIONS

#### 3.1 *Introduction*

We discuss in this chapter a natural extension of Kuratowski's Theorem with applications to matching theory and the study of Pfaffian orientations. As mentioned in Chapter 1, Kuratowski's Theorem is a fundamental result at the heart of our understanding of planar graphs:

**Theorem 3.1.1.** *A graph is planar if and only if it does not contain  $K_5$  or  $K_{3,3}$  as a subdivision.*

It turns out that  $K_{3,3}$  is the more important of these in some sense; for a sufficiently well-connected large graph, a classical result allows us to ignore  $K_5$ :

**Theorem 3.1.2.** *A 3-connected graph is non-planar if and only if it either contains a subgraph isomorphic to  $K_5$  or contains  $K_{3,3}$  as a subdivision.*

In the case of well-connected non-planar bipartite graphs, this raises an interesting question: when do such graphs contain  $K_{3,3}$  in a bipartite way? We formalize this as follows.

**Definition.** Let  $G$  be a graph and  $H$  a subgraph of  $G$  isomorphic to a subdivision of  $K_{3,3}$ . Let  $v_1, v_2, \dots, v_6$  be the degree three vertices of  $H$  and for  $i = 1, 2, 3$  and  $j = 4, 5, 6$  let  $P_{ij}$  be the paths in  $H$  between  $v_i$  and  $v_j$ . We then refer to  $H$  as a *hex* or a *hex of  $G$* , the vertices  $v_i$  as the *feet* of  $H$ , and the paths  $P_{ij}$  as the *segments* of  $H$ . A segment is *odd* if it has an odd number of edges, and *even* otherwise. A hex  $H$  is *odd* if every segment of  $H$  is odd.

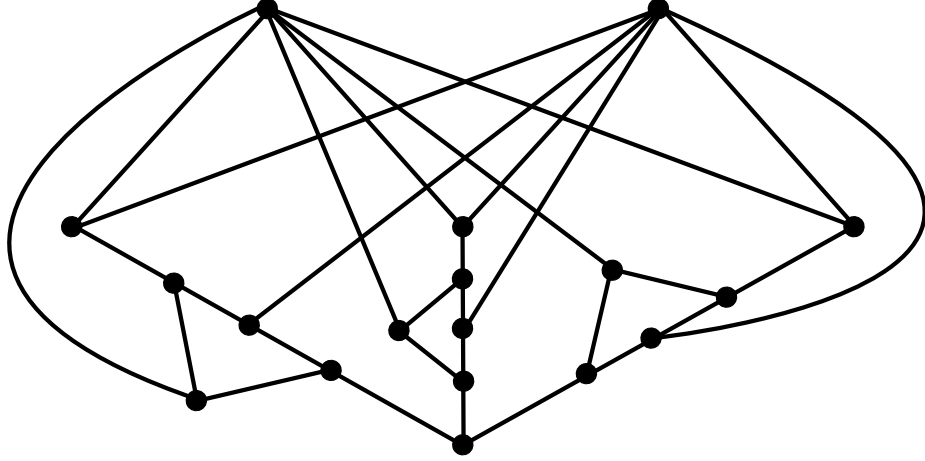


Figure 3.1: A graph showing that Theorem 3.1.3 does not extend to 3-connected graphs

So our question is, given a non-planar bipartite graph, when does it contain an odd hex? If  $G$  is a bipartite graph with bipartition  $(A, B)$ , this is equivalent to finding three vertices of  $A$ , three vertices of  $B$  and the appropriate 9 vertex-disjoint paths between them. In a sense, then, this is a natural way to find one bipartite graph inside another. The answer to this question is the main theorem of this section:

**Theorem 3.1.3.** *Every internally 4-connected bipartite non-planar graph has an odd hex.*

We say that a bipartite graph  $G$  is *internally 4-connected* if it is 3-connected, has at least five vertices, and there is no partition  $(A, B, C)$  of  $V(G)$  such that  $|A|, |B| \geq 2$ ,  $|C| = 3$  and  $G$  has no edge with one end in  $A$  and the other in  $B$ . This theorem is not true in the case of 3-connected bipartite graphs; a counterexample is depicted in Figure 3.1. Similarly it is false in the case of non-bipartite graphs; a counterexample is the graph  $V_8$  which consists of an 8-cycle along with the 4 chords between vertices of distance 4 along the cycle. This graph is depicted in Figure 3.2.

In addition to being a natural question to ask about the structure of non-planar graphs, this theorem also has applications in matching theory. In Chapter 4, we will

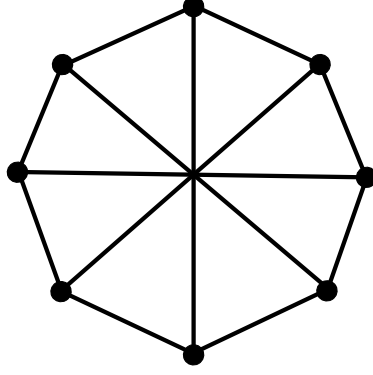


Figure 3.2: The graph  $V_8$

take advantage of this result to provide a different proof of a nice result of Robertson, Seymour, and Thomas found in [42].

### 3.2 Lemmas

In this section we prove two lemmas that we will need for the proof of Theorem 3.1.3.

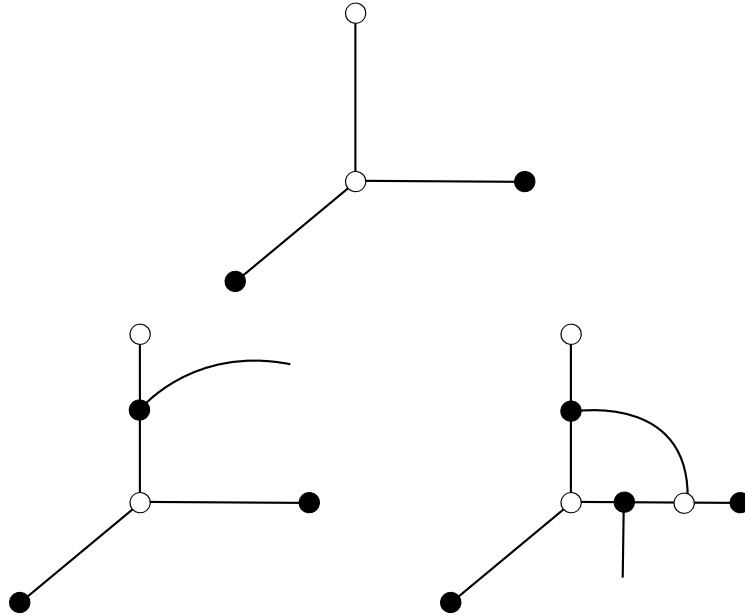


Figure 3.3: The structures of Lemma 3.2.1

**Lemma 3.2.1.** *Let  $G$  be an internally 4-connected bipartite graph with bipartition  $(A, B)$ . Let  $a, v \in A$  and  $b, c \in B$  with paths  $P_1 = v \dots a, P_2 = v \dots b, P_3 = v \dots c$*

vertex disjoint except for  $v$ . Let  $X \subseteq V(G)$  with  $|X| \geq 2$  be disjoint from  $V(P_1) \cup V(P_2) \cup V(P_3)$ . Then at least one of the following holds:

- (1) There exist  $v' \in A, u \in B, x \in X$  and paths  $P'_1 = v' \dots a, P'_2 = v' \dots b, P'_3 = v' \dots c, P'_4 = u \dots x$  such that  $u \in V(P'_1)$  and all of the  $P'_i$  are vertex disjoint and are disjoint from  $X$  except that  $v' \in V(P'_1) \cap V(P'_2) \cap V(P'_3), u \in V(P'_1) \cap V(P'_4)$  and  $x \in X \cap V(P'_4)$ .
- (2) There exists  $v', s \in A, u, t \in B, x \in X$  and paths  $P'_1 = v' \dots a, P'_2 = v' \dots t \dots s \dots b, P'_3 = v' \dots c, P'_4 = u \dots s, P'_5 = t \dots x$  such that  $u \in V(P'_1)$  and all of the  $P'_i$  are vertex disjoint and are disjoint from  $X$  except that  $v' \in V(P'_1) \cap V(P'_2) \cap V(P'_3), u \in V(P'_1) \cap V(P'_4), s \in V(P'_2) \cap V(P'_4), t \in V(P'_2) \cap V(P'_5), x \in V(P'_5) \cap X$ .

**Definition.** We will refer to the paths  $P'_1, P'_2$ , and  $P'_3$  as the *replacement paths* and the paths  $P'_4$  and, when appropriate,  $P'_5$  as the *new paths*. In the forthcoming arguments we will apply either the induction hypothesis or Lemma 3.2.1 to various carefully selected paths  $R_1, R_2, R_3$  to obtain replacement paths  $R'_1, R'_2, R'_3$  and new paths  $R'_4$  and, when appropriate,  $R'_5$ . However, we will be able to assume that  $R'_1 = R_1, R'_2 = R_2$  and  $R'_3 = R_3$  which will simplify our notation. We will refer to this assumption as assuming that the *replacement paths do not change*.

*Proof of Lemma 3.2.1.* Let the paths  $P_1, P_2, P_3$  be fixed. By an *augmenting sequence* we mean a sequence of paths  $Q_1, \dots, Q_k$ , where the ends of  $Q_i$  are  $v_{2i-1}$  and  $v_{2i}$ ,  $v_{2k} \in X, v_1 \in V(P_1) - \{a, v\}$ , each other  $v_i$  is in  $P_j \setminus v$  for some  $j \in \{1, 2, 3\}$ , and all the  $Q_i$  are vertex disjoint from one another and disjoint from the  $P_i$  and  $X$  except for their ends. Further, for  $j > 1$  odd,  $v_j$  and  $v_{j-1}$  are distinct and both lie on the same  $P_i$  and  $v_j$  lies between  $v$  and  $v_{j-1}$  on  $P_i$ . If  $v_i, v_j \in V(P_l)$  and  $i < j - 1$ , then  $v, v_i, v_j$  appear in  $P_l$  in the order listed (possibly  $v_i = v_j$ ). We refer to each  $Q_i$  as an *augmentation*. The *length* of an augmenting sequence is the number of

augmentations it has. We define the *index* of the augmenting sequence  $Q_1, Q_2, \dots, Q_k$  to be the smallest integer  $i$  such that either  $i$  is odd and  $v_i \in A$ , or  $i$  is even and  $v_i \in B$ , or  $i = 2k + 1$ .

We proceed by induction on the size of  $V(G) - X$ . Since  $G$  is internally 4-connected it follows by the standard “augmenting path” argument from network flow theory or from Lemmas 3.3.2 and 3.3.3 in [13] that there exists an augmenting sequence.

Choose the vertex  $v$ , paths  $P_1, P_2, P_3$ , and an augmenting sequence  $S = (Q_1, \dots, Q_k)$  such that the length of  $S$  is as small as possible, and, subject to that, the index of  $S$  is as large as possible. Let  $v_1, v_2, \dots, v_{2k}$  be the ends of the paths  $Q_i$ , numbered as above. Then it follows that for  $j = 2, 4, \dots, 2k - 2$  the vertex  $v_j$  lies on a different path  $P_i$  than  $v_{j-1}$ . Note that this lemma is equivalent to showing that the length of  $S$  is at most 2 and that the index of  $S$  is at least twice the length of  $S$ .

Suppose first that the length of  $S$  is 1. Then we may assume that the index of  $S$  is 1, so  $v_1 \in A$ . Let  $X' = X \cup V(v_1 P_1 a \cup Q_1 \setminus v_1)$ . Then apply the induction hypothesis to the paths  $v P_1 v_1, P_2, P_3$  and set  $X'$ . We may assume that the replacement paths do not change. Suppose we have outcome (1). Thus there exists a path  $P_4$  with ends  $u \in B \cap V(v P_1 v_1)$  and  $x \in X'$ , disjoint from  $V(P_1 \cup P_2 \cup P_3) \cup X'$ , except for its ends. If  $x \in X$ , then this is exactly outcome (1) in the original situation. If  $x \in V(Q_1)$ , then take  $P'_4 = P_4 \cup x Q_1 v_2$  to find outcome (1) in the original situation. If  $x \in V(v_1 P_1 a) - \{v_1\}$ , then take  $P'_1 = v P_1 u \cup P'_4 \cup x P_1 a$ ,  $P'_4 = u P_1 v_1 \cup Q_1$  to again have outcome (1). So we must have outcome (2), and so there exist paths  $P_4, P_5$  as stated in (2). Again, if  $x \in X$ , this is exactly outcome (2), and if  $x \in V(Q_1)$ , then taking  $P'_5 = P_5 \cup x Q_1 v_2$  again gives outcome (2). So we may assume  $x \in V(v_1 P_1 a)$ . We then take  $v' = v$ ,  $P'_1 = v P_2 t P_5 x P_1 a$ ,  $P'_2 = v P_1 u P_4 s P_2 b$ ,  $P'_3 = v P_3 c$ ,  $P'_4 = t P_2 s$ ,  $P'_5 = u P_1 v_1 Q_1 v_2$ , which is an instance of outcome (2).

So we may assume that the length of  $S$  is at least 2. Suppose that the index of  $S$  is 1, so  $v_1 \in A$ . Without loss of generality, we may assume that  $v_2 \in V(P_2)$ .



Let  $u \in B$  lie on  $P'_1$  between  $v_1$  and  $a$ . Since  $\{a, v_1\}$  is not a 2-separation in  $G$ , we can apply Menger's Theorem to find three paths from  $u$ , one to  $a$ , one to  $v_1$  and one to  $V(X) \cup V(Q_1) \cup V(Q_2) \cup V(vP_1v_1) \cup V(P_2) \cup V(P_3)$  labeled  $R_1, R_2, R_3$  respectively. We replace  $v_1P_1a$  by  $R_1 \cup R_2$  and simply refer to  $R_3$  as  $R$ . Let the ends of  $R$  be  $u$  and  $r$ . If  $r \in X$ , then we have an augmenting sequence of length 1 contrary to the choice of  $S$ . If  $r \in V(Q_i)$ , then we have found an augmenting sequence of at most the same length as  $S$ , but with index at least 2. If  $r \in V(P_1)$ , then we take  $P'_1 = vP_1rRuP_1a$  and  $Q'_1 = uP_1v_1Q_1v_2$ , which gives an augmenting sequence of the same length but with higher index. If  $r$  is on  $P_2$  between  $v_3$  and  $b$  with  $r \neq v_3$ , then taking  $Q'_1 = R$  is immediately an augmenting sequence with the same length and higher index. If  $r$  is on  $P'_2$  between  $v_3$  and  $v'$ , then let  $v' = v_1$ ,  $P'_1 = v_1P_1a$ ,  $P'_2 = v_1Q_1v_2P_2b$ ,  $P'_3 = v_1P_1vP_3c$ , and  $Q_1 = uRrP_2v_3Q_2v_4$  which gives an augmenting sequence of shorter length. Similarly, if  $r$  is on  $P'_3$ , let  $v' = v_1$ ,  $P'_1 = v_1P_1a$ ,  $P'_2 = v_1Q_1v_2P_2b$ ,  $P'_3 = v_1P_1vP_3c$ , and  $Q'_1 = R$ ,  $Q'_2 = vP_2v_3Q_2v_4$  which is an augmenting sequence of the same length but higher index.

So we may assume that the index of  $S$  is at least 2. Suppose the index is exactly 2, so  $v_2 \in B$ . Then apply the induction hypothesis to the paths  $Q_1$ ,  $v_1P_1a$  and  $v_1P_1v$  and set  $X' := X \cup V(P_2) \cup V(P_3) \cup V(Q_2) \cup V(Q_3) \cup \dots \cup V(Q_k) - \{v, v_2\}$ . We assume, since we may, that the replacement paths do not change. Suppose we have outcome (1). This gives a vertex  $u \in A$  on  $Q_1$  and  $x$  in  $X'$  with a path  $P_4$  between them. If  $x \in X$ , then take  $u' = v_1$ ,  $P'_4 = v_1Q_1uP_4x$  to get outcome (1). If  $x \in Q_i$  for  $i > 1$ , then take  $Q'_1 = v_1Q_1uP_4xQ_i$  which gives the shorter augmenting sequence  $Q'_1, Q_{i+1}, Q_{i+2}, \dots, Q_k$ . If  $x$  is on  $P_2$  between  $v_2$  and  $b$ , then take  $P'_2 = vP_1v_2Q_1uP_4xP_2b$ ,  $v'_2 = u$ , and  $Q'_1 = v_1Q_1u$  which gives an augmenting sequence of higher index. If  $x$  is on  $P_2$  between  $v$  and  $v_2$ , then we take  $P'_2 = vP_2xP_4uQ_1v_2P_2b$ ,  $v'_3 = x$ , and  $Q'_2 = xP_2v_3Q_2v_4$  if  $v_3 \in v_2P_2x$  and  $Q'_2 = Q_2$  if  $v_3 \in xP_2v$  which gives an augmenting sequence with

higher index. Finally, if  $x$  is on  $P_3$ , then take  $v' = u, P'_1 = uQ_1v_1P_1a, P'_2 = uQ_1v_2P_2b, P'_3 = uP_4xP_3c$ , and  $Q'_1 = v_1P_1vP_2v_3Q_2v_4$  which gives a shorter augmenting sequence.

So instead we have outcome (2) and we use the notation for  $u, s, t, x, P_4, P_5$  as listed in the outcome. It follows that  $P_1 = vP_1sP_1tP_1v_1P_1a$ . If  $x \in X$ , then taking  $P'_1 = vP_1sP_4uQ_1v_1P_1a, P'_2 = P_2, P'_3 = P_3, P'_4 = sP_1tP_5x$  gives outcome (1). If  $x \in V(P'_2)$  between  $v_2$  and  $b$ , we take  $v' = t, P'_1 = tP_1a, P'_2 = tP_5xP_2b, P'_3 = tP_1vP_3c, Q'_1 = v_1Q_1v_2P_2v_3Q_2v_4$  and then  $Q'_1, Q_3, Q_4, \dots, Q_k$  is an augmenting sequence of length  $k - 1$ , a contradiction. If  $x$  is on  $P'_2$  between  $v$  and  $v_2$ , take  $v' = u, P'_1 = uQ_1v_1P_1a, P'_2 = uQ_1v_2P_2b, P'_3 = uP_4sP_1vP_3c, Q'_1 = v_1P_1tP_5xP_2v_3Q_2v_4$  which gives a shorter augmenting sequence. If  $x \in V(Q_i)$  with  $i > 1$ , then taking  $Q'_1 = tP_5xQ_iv_{2i}$  gives the augmenting sequence  $Q'_1, Q_{i+1}, Q_{i+2}, \dots, Q_k$  which contradicts the choice of  $S$ . Finally, if  $x$  is on  $P'_3$ , take  $v' = t, v'_2 = u, v'_3 = s, P'_1 = tP_1a, P'_2 = tP_1sP_4uQ_1v_2P_2b, P'_3 = tP_5xP_3c, Q'_1 = v_1Q_1u$ , and  $Q'_2 = sP_1vP_2v_3Q_2$  to get an augmenting sequence of the same length and higher index.

So we may assume the index of  $S$  is at least 3. Suppose the index is exactly 3, so  $v_3 \in A$ . Note that  $v_2$  and  $v'$  are completely symmetric with respect to this augmenting sequence (up to  $v_3$ ). We apply induction to the paths  $v_2P_2v_3, Q_1, v_2P_2b$  and set  $X' := X \cup V(P_1 \cup vP_2v_3 \cup P_3 \cup Q_2 \cup Q_3 \cup \dots \cup Q_k) - \{v_1, v_3\}$ . Since we may, we assume the replacement paths do not change. Suppose first we find outcome (2). We use the notation in the outcome for  $u, s, t, x$ . If  $x \in X$ , then taking  $P'_4 = v_1Q_1tP_5x$  with  $u' = v_1$  gives outcome (1). If  $x$  is on  $Q_i, i > 1$ , we take  $Q'_1 = v_1Q_1tP_5xQ_i$  which gives a shorter augmenting sequence. If  $x$  is on  $P_2$  between  $v$  and  $v_3, P_3$ , or on  $P_1$  between  $v_1$  and  $v$ , let  $i$  be such that  $P_i$  contains  $x$ , then we can take  $v' = v_2, P'_1 = v_2P_2uP_4sQ_1v_1P_1a, P'_2 = v_2P_2b, P'_3 = v_2Q_1tP_5xP_iP_3c, v'_1 = u, Q'_1 = uP_2v_3Q_2v_4$  to find a shorter augmenting sequence. Finally, if  $x$  is on  $P_1$  between  $v_1$  and  $a$ , we take  $v' = v_2, P'_1 = v_2Q_1tP_5xP_1a, P'_2 = v_2P_2b, P'_3 = v_2P_2uP_4sQ_1v_1P_1vP_3c, v'_1 = t, v'_2 = s, v'_3 = u, Q'_1 = tQ_1s, Q'_2 = uP_2v_3Q_2v_4$  which gives an augmenting sequence with the

same length and higher index.

So instead we consider outcome (1) with the notation for  $u, x, P_4$  as in the outcome. If  $x \in X$ , we can take  $u' = v_1, s' = v_2, t' = u, P'_4 = Q_1, P'_5 = P_4$  to find outcome (2). If  $x$  is on a  $Q_i, i > 1$ , then we take  $v'_3 = u, Q'_2 = uP_4xQ_i$  which gives an augmenting sequence of at most the same length but higher index. If  $x$  is on  $P_2$  between  $v'$  and  $v_3$ , then take  $P'_2 = vP_2xP_4uP_2b, v'_3 = u, Q'_2 = uP_2v_3Q_2v_4$  to find an augmenting sequence of the same length with higher index. If  $x$  is on  $P_3$ , we take  $v' = v_2, P'_1 = v_2Q_1v_1P_1a, P'_2 = v_2P_2b, P'_3 = v_2P_2uP_4xP_3c, Q'_1 = v_1P_2vP_2v_3Q_2v_4$  which gives a shorter augmenting sequence. If  $x$  is on  $P_1$  between  $v_1$  and  $a$ , then take  $v' = v_2, P'_1 = v_2P_2uP_4xP_1a, P'_2 = v_2P_2b, P'_3 = v_2Q_1v_1P_1vP_3c, v'_1 = u, Q'_1 = uP_2v_3Q_2v_4$  which gives a shorter augmenting sequence. Finally, suppose  $x$  is on  $P_1$  between  $v$  and  $v_1$  and  $x \in A$ . Then consider  $P'_1 = v_2Q_1v_1P_1a, P'_2 = v_2P_2b, P'_3 = v_2P_2uP_4xP_1vP_3c, Q'_1 = v_1P_1x, Q'_2 = uP_2v_3Q_2$  which gives an augmenting sequence of the same length and lower index.

So we must have  $x \in B$  on  $P_1$  between  $v$  and  $v_1$ . We apply induction to the paths  $vP_2v_3, vP_1x, P_3$  and set  $X' := X \cup V(xP_1a \cup v_3P_2b \cup P_4 \cup Q_1 \cup Q_2 \cup Q_3 \cup \dots \cup Q_k) - \{x, v_3\}$ . This gives us  $u_2 \in A$  between  $v$  and  $v_3$  with a path  $P_5$  to  $x_2$ . Suppose we have outcome (1). Then if  $x_2$  is not on  $Q_1, P_4$ , or  $xP_1v_1$ , then by symmetry we can apply the analysis of the previous paragraph. If  $x_2$  is on  $P_4$ , then we replace  $P_4$  with  $uP_4x_2P_5u_2$  and have one of the outcomes above. If  $x_2$  is on  $Q_1$ , then take  $v' = v, P'_1 = P_1, P'_2 = vP_2u_2P_5x_2Q_1v_2P_2b, P'_3 = P_3, v'_1 = x, Q'_1 = xP_4uP_2v_3Q_2v_4$  to find a shorter augmenting sequence. Finally, if  $x_2$  is on  $xP_1v_1$ , then take  $v' = v_2, P'_1 = v_2Q_1v_1P_1a, P'_2 = v_2P_2b, P'_3 = v_2P_2uP_4xP_1vP_3c, Q'_1 = v_1P_1x_2P_5u_2P_2v_3Q_2$  which is a shorter augmenting sequence. So we must have outcome (2) of the lemma which gives vertices  $u_2, s, t, x_2$  and paths  $P_5$  and  $P_6$ . If  $x_2$  is not on either  $P_4$  or  $v_1P_1x$ , then we can apply the analysis from the previous two paragraphs. Suppose  $x_2 \in V(P_4)$ . Then take  $v' = v, P'_1 = vP_2u_2P_5sP_1a, P'_2 = vP_1tP_6x_2P_4uP_2b, P'_3 = P_3, Q'_1 = u_2P_2v_3Q_2$  to get

a shorter augmenting sequence. Finally, suppose  $x_2 \in V(v_1 P_1 x)$ . Then take  $v' = s$ ,  $P'_1 = s P_1 t P_6 x_2 P_1 a$ ,  $P'_2 = s P_1 x P_4 u P_2 b$ ,  $P'_3 = s P_5 u_2 P_2 v P_3 c$ ,  $Q'_1 = Q$ ,  $Q'_2 = u P_2 v_3 Q_2$  which is an augmenting sequence of the same length and lower index.

So we may finally assume that the length of  $S$  is at least 3 with index at least 4. Note that  $v_4$  must be on  $P'_3$  (still assuming that  $v_2$  was on  $P'_2$ ), since otherwise we get a shorter augmenting sequence. But then take  $v' = v_2$ ,  $P'_1 = v_2 Q_1 v_1 P_1 a$ ,  $P'_2 = v_2 P_2 b$ ,  $P'_3 = v_2 P_2 v_3 Q_2 v_4 P_3 c$ ,  $Q'_1 = v_1 P_1 v P_3 v_5 Q_3 v_6$ , which gives a shorter augmenting sequence.  $\square$

Lemma 3.2.1 will suffice for most of our arguments. However, on one occasion we will need the following strengthening.

**Lemma 3.2.2.** *Let  $G$  be an internally 4-connected bipartite graph with bipartition  $(A, B)$ . Let  $a, v \in A$  and  $b, c \in B$  with paths  $P_1 = v \dots a$ ,  $P_2 = v \dots b$ ,  $P_3 = v \dots c$  vertex disjoint except for  $v$ . Let  $X \subseteq V(G)$  be disjoint from  $V(P_1) \cup V(P_2) \cup V(P_3)$ . Then at least one of the following holds:*

- (A) *There exist vertices  $v' \in A, u \in B, x \in X \cap A$  and paths  $P'_1 = v' \dots a$ ,  $P'_2 = v' \dots b$ ,  $P'_3 = v' \dots c$ ,  $P'_4 = u \dots x$  such that  $u \in V(P'_1)$  and all of the  $P'_i$  are vertex disjoint and are disjoint from  $X$  except as specified,*
- (B) *There exist vertices  $v', s \in A, u, t \in B, x \in X \cap B$  and paths  $P'_1 = v' \dots a$ ,  $P'_2 = v' \dots b$ ,  $P'_3 = v' \dots c$ ,  $P'_4 = u \dots s \dots x$ ,  $P'_5 = s \dots t$  such that  $u \in V(P'_1)$ ,  $t \in V(X \cup P'_2 \cup P'_3)$ , and all of the  $P'_i$  are vertex disjoint and are disjoint from  $X$  except as specified and except that  $t$  may lie on  $P'_2$  or  $P'_3$ ,*
- (C) *There exist vertices  $v', s \in A, u, t \in B, x \in X$  and paths  $P'_1 = v' \dots a$ ,  $P'_2 = v' \dots t \dots s \dots b$ ,  $P'_3 = v' \dots c$ ,  $P'_4 = u \dots s$ ,  $P'_5 = t \dots x$  such that  $u \in V(P'_1)$  and all of the  $P'_i$  are vertex disjoint and are disjoint from  $X$  except as specified,*

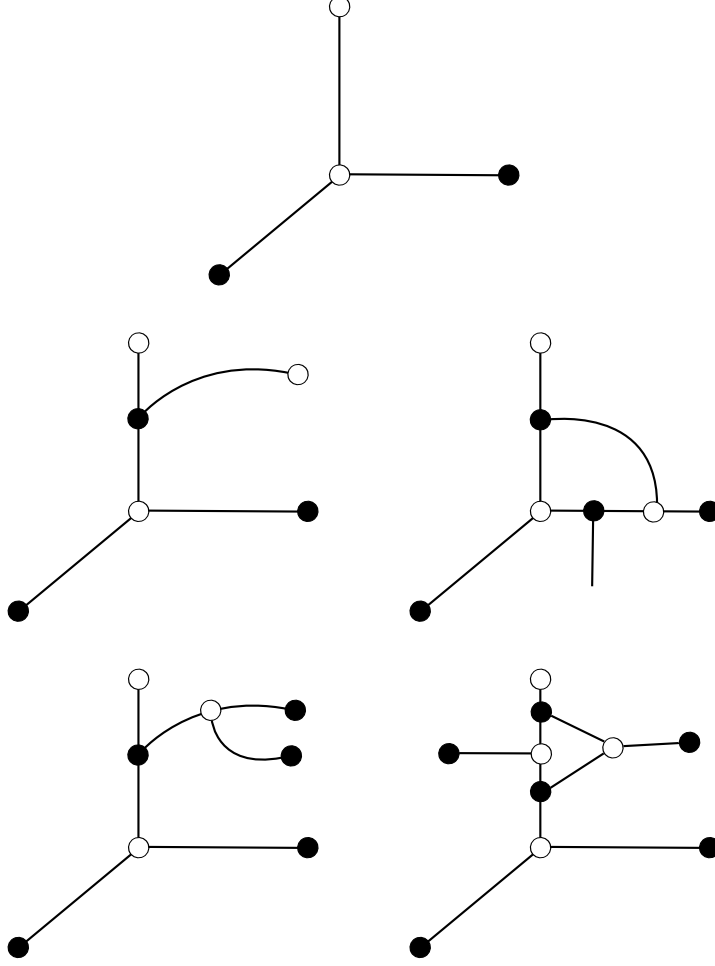


Figure 3.4: The structures of Lemma 3.2.2

(D) *There exist vertices  $v', s, w \in A, u, t \in B, x, y \in X \cap B$  and paths  $P'_1 = v' \dots u \dots w \dots t \dots a$ ,  $P'_2 = v' \dots b$ ,  $P'_3 = v' \dots c$ ,  $P'_4 = u \dots s \dots x$ ,  $P'_5 = s \dots t$ ,  $P'_6 = w \dots y$  such that all of the  $P'_i$  are vertex disjoint and are disjoint from  $X$  except as specified and except that  $x$  may equal  $y$ .*

*Proof.* We proceed by induction on the size of  $|V(G)| - |X|$ . Apply Lemma 3.2.1. We may assume that we are in the first outcome of that lemma since the second outcome is outcome (C) of this lemma. We may assume that the replacement paths do not change. Thus there exists a path  $P_4$  with ends  $u \in B \cap V(P_1)$  and  $x \in X$ , vertex-disjoint from  $P_1 \cup P_2 \cup P_3$ , except for  $u$ . We may assume that  $x \in B$ , for

otherwise (A) holds.

We apply the induction hypothesis to the paths  $P_4, uP_1a$  and  $uP_1v$ , and set  $X' = X \cup V(P_2 \cup P_3 \setminus \{x, v\})$ . Note that  $|X'| > |X|$  since  $b$  and  $c$  are distinct, not in  $X$ , and in  $P_2 \cup P_3$  and  $v$  was not in  $X$  originally. We consider each of the four outcomes separately. We may assume that the replacement paths do not change.

If outcome (A) holds, then we obtain outcome (B) of the lemma. Next, let us assume that the induction hypothesis yields outcome (B). Thus there exist vertices  $s \in A \cap V(P_4)$ ,  $t \in B$ ,  $y \in A \cap (V(P_2 \cup P_3) \cup X)$  and  $w \in A \cap (V(P_1 \cup P_2 \cup P_3) \cup X)$ , and paths  $P_5$  from  $s$  to  $y$  and  $P_6$  from  $t$  to  $w$  such that the paths  $P_5$  and  $P_6$  are disjoint and disjoint from  $V(P_1 \cup P_2 \cup P_3) \cup X$ , except as stated. If  $y \in X$ , then we have the outcome (A) by taking  $P'_1 = P_1, P'_2 = P_2, P'_3 = P_3, P'_4 = uP_4sP_5y$  and  $v' = v, u = u, x = y$ . So without loss of generality, we may assume  $y \in V(P_2)$ . We are now interested in where  $w$  lies. If  $w$  lies on  $P_2$ , then we may assume that it lies between  $y$  and  $b$  since  $y$  and  $w$  are then symmetric. In that case, take  $P'_1 = P_1, P'_2 = vP_2yP_5tP_6wP_2b, P'_3 = P_3, P'_4 = P_4, P'_5 = sP_5t, x = x, t = t, s = s, u = u, v' = v$  which is exactly outcome (B). If  $w$  lies on  $P_1$  between  $v$  and  $u$ , take  $v' = w, s = s, t = t, u = u, x = x, P_1 = wP_1a, P_2 = wP_6tP_5yP_2b, P_3 = wP_1vP_3c, P_4 = uP_4x, P_5 = sP_5t$  which is again outcome (B). If  $w$  lies on  $P_3$ , , take  $v' = w, s = s, t = t, u = u, x = x, P_1 = wP_3vP_1a, P_2 = wP_6tP_5yP_2b, P_3 = wP_3c, P_4 = uP_4x, P_5 = sP_5t$ , which is again outcome (B). So we have that  $w$  lies on  $P_1$  between  $u$  and  $a$  (or is  $a$ ).

Let  $r \in B$  lie between  $v$  and  $y$  on  $P_2$ . By Menger's theorem and by replacing  $vP_2y$  if necessary we may assume that there exists a path  $P_7$  from  $r$  to a vertex  $z$  not on  $vP_2y$  that is disjoint from  $P_1 \cup \dots \cup P_6$ , except for its ends. If  $z \in X$ , then keeping  $v = v', P'_1 = P_1, P'_2 = P_2, P'_3 = P_3$  and taking  $u = u, t = r, s = y, x = z, P'_4 = uP_4sP_5y, P'_5 = P_7$ , this is outcome (C). If  $z \in V(P_1)$  between  $v$  and  $u$  (note that this is symmetric with  $z \in V(P_3)$ ), then take  $v' = y, u = t, s = s, t = r, x = x, P'_1 = yP_5tP_6wP_1a, P'_2 = yP_2b, P'_3 = yP_2vP_3c, P'_4 = tP_5sP_4x, P'_5 = sP_4uP_1zP_7r$  to find

outcome (B). If  $z$  is on  $P_1$  between  $u$  and  $w$ , take  $v' = y, u = t, s = s, t = u, x = x, P'_1 = yP_5tP_6wP_1a, P'_2 = yP_2b, P'_3 = yP_3rP_7zP_1vP_3c, P'_4 = tP_5sP_4x, P'_5 = sP_4u$ , which is outcome (B). If  $z$  is on  $P_1$  between  $w$  and  $a$ , take  $v' = y, u = r, t = u, s = v, x = x, P'_1 = yP_2rP_7zP_1a, P'_2 = yP_2b, P'_3 = yP_5tP_6wP_1vP_3c, P'_4 = rP_2a, P'_5 = uP_4x$ , which gives outcome (C). If  $z$  is on  $P_2$  between  $y$  and  $b$ , then take  $v = v', P'_1 = P_1, P'_2 = vP_2rP_7zP_2b, P'_3 = P_3, u = u, s = s, x = x, t = r, P'_4 = P_4, P'_5 = sP_5yP_2r$  to again find outcome (B). If  $z$  is on  $P_4$  between  $u$  and  $s$ , take  $v' = y, u = t, s = s, t = r, x = x, P'_1 = yP_5tP_6wP_1a, P'_2 = yP_2b, P'_3 = yP_2vP_3c, P'_4 = tP_5sP_4x, P'_5 = sP_4zP_7r$  which is outcome (B). If  $z$  is on  $P_4$  between  $s$  and  $x$ , take  $v' = v, u = u, s = y, t = r, P'_1 = P_1, P'_2 = P_2, P'_3 = P_3, P'_4 = uP_4sP_5y, P'_5 = rP_7zP_4x$  to get outcome (C). If  $z$  is on  $P_5$  or  $P_6$ , then take  $v = v', P'_1 = P_1, P'_2 = P_2, P'_3 = P_3, u = u, s = s, x = x, t = r, P'_4 = P_4$ . If  $z$  is on  $P_5$ , then take  $P_5 = sP_5zP_7r$  to find outcome (B) and if  $z$  is on  $P_6$ , take  $P_5 = sP_5tP_6zP_7r$  to find outcome (B). This completes the case when induction yields outcome (B).

Next we assume that induction yields outcome (C). Thus there exist vertices  $s \in A \cap V(P_4), t \in B \cup V(P_1), w \in A \cap V(P_1)$  and  $y \in X'$ , and paths  $P_5, P_6$  such that  $v, u, w, t, a$  occur on  $P_1$  in the order listed,  $P_5$  has ends  $s$  and  $t$ ,  $P_6$  has ends  $w$  and  $y$ , and  $P_5, P_6$  are disjoint and disjoint from  $P_1, P_2, P_3, P_4$ , except for their ends. Note that if  $y \in X \cup B$ , this is exactly outcome (D), including the notation. Suppose first that  $y \in X \cup A$ . Then take  $v' = v, u = u, x = y, P'_1 = vP_1uP_4sP_5tP_1a, P'_2 = P_2, P'_3 = P_3, P'_4 = uP_1wP_6y$  to find outcome (A). So we may assume that  $y \in V(P_2)$ . Then take  $v' = w, u = t, s = s, t = u, x = x, P'_1 = wP_1a, P'_2 = wP_6yP_2b, P'_3 = wP_1vP_3c, P'_4 = tP_5sP_4x, P'_5 = sP_4u$ , which is outcome (C). Since  $y \in V(P_3)$  is symmetric with this case, that completes this outcome.

Finally, we assume that induction yields outcome (D). Thus there exist vertices  $s, t \in A \cap V(P_4), r \in B \cap V(P_4), w \in B$  and  $y, z \in A \cap X'$ , and paths  $P_5, P_6, P_7$  such that  $u, s, r, t, x$  occur on  $P_4$  in the order listed,  $P_5$  has ends  $s$  and  $y$  and includes

$w$ ,  $P_6$  has ends  $w$  and  $t$ ,  $P_7$  has ends  $r$  and  $z$ , and the paths  $P_5, P_6, P_7$  are pairwise disjoint and disjoint from  $P_1, P_2, P_3, P_4$ , except for their ends. Note that  $y$  and  $z$  are completely symmetric as are  $r$  and  $w$ . Suppose that  $y \in X$ . Then take  $v' = v$ ,  $u = u, x = y, P'_1 = P_1, P'_2 = P_2, P'_3 = P_3, P'_4 = uP_4sP_5y$  to get outcome (A). So we may assume  $y \in V(P_2)$ . Suppose  $z$  lies on  $P_2$  between  $v$  and  $y$  (by symmetry, if  $z$  lies on  $P_2$ , this assumption is without loss of generality). Then take  $v' = v, u = u, t = r, x = x, s = s, P'_1 = P_1, P'_2 = vP_2zP_7rP_4sP_5yP_2b, P'_3 = P_3, P'_4 = uP_4s, P'_5 = rP_4x$  which is outcome (C). So  $z$  lies on  $P_3$ . Then take  $v' = y, u = u, s = s, t = r, x = x, P'_1 = yP_2vP_1a, P'_2 = yP_2b, P'_3 = yP_5sP_4rP_7zP_3c, P'_4 = uP_4s, P'_5 = rP_4x$  which is again outcome (C). This completes this outcome and the proof.  $\square$

### 3.3 Proof of Theorem 3.1.3

Let  $H$  be a subgraph of a graph  $G$ . By an  $H$ -path in  $G$  we mean a path in  $G$  with at least one edge, both ends in  $V(H)$  and no other vertex or edge in  $H$ .

Let  $H$  be a hex in a graph  $G$ , let  $P$  be the union of a set of  $H$ -paths in  $G$ , and let  $Q$  be a subgraph of  $H$ . We denote by  $H + P - Q$  the graph obtained from  $H \cup P$  by deleting all edges of  $Q$  and then deleting all resulting isolated vertices. A typical application will be when  $P$  and  $Q$  are paths, but we will need more complicated choices.

A hex in a graph  $G$  is *optimal* if no hex of  $G$  has strictly more odd segments. We proceed by a series of lemmas, each improving a lower bound on the number of odd segments in an optimal hex.

**Lemma 3.3.1.** *Let  $G$  be a 3-connected bipartite graph. Then every optimal hex of  $G$  has at least four odd segments.*

*Proof.* Let  $(A, B)$  be a bipartition of  $G$  and let  $H$  be an optimal hex in  $G$  with feet and segments numbered as in the definition of a hex. We may assume for a contradiction that  $H$  has at most three odd segments. It follows that at least five feet of  $H$  belong



to the same set  $A$  or  $B$ , and so we may assume that  $v_1, v_2, \dots, v_5$  all belong to  $A$ . Thus  $P_{14}$  has an internal vertex  $u$  that belongs to  $B$ . Since  $G$  is 3-connected we may assume, by replacing  $P_{14}$  if necessary, that there exists an  $H$ -path  $Q$  with one end  $u$  and the other end, say  $w$ , in  $V(H) - V(P_{14})$ . By symmetry, we may assume that  $w$  belongs to  $P_{15}, P_{16}, P_{24}, P_{25}$ , or  $P_{26}$ . Let  $R$  be defined as  $v_1P_{15}w$ ,  $v_1P_{16}w$ ,  $v_4P_{24}w$ ,  $P_{24}$  or  $P_{24}$ , respectively. Then  $H + Q - R$  is a hex with strictly more odd segments than  $H$ , contrary to the optimality of  $H$ .  $\square$

**Lemma 3.3.2.** *Let  $G$  be an internally 4-connected bipartite graph. Then every optimal hex of  $G$  has at least five odd segments.*

*Proof.* Let  $(A, B)$  be a bipartition of  $G$  and let  $H$  be an optimal hex in  $G$  with feet and segments numbered as in the definition of a hex. By Lemma 3.3.1 we may assume for a contradiction that  $H$  has exactly four odd segments. It follows that two feet of  $H$  in  $\{v_1, v_2, v_3\}$  and two feet of  $H$  in  $\{v_4, v_5, v_6\}$  belong to the same set  $A$  or  $B$ , and so we may assume that  $v_1, v_2, v_4, v_5$  all belong to  $A$ . Thus  $P_{14}$  has an internal vertex  $u$  that belongs to  $B$ . Since  $G$  is 3-connected we may assume, by replacing  $P_{14}$  if necessary, that there exists an  $H$ -path  $Q$  with one end  $u$  and the other end, say  $w$ , in  $V(H) - V(P_{14})$ . By symmetry, we may assume that  $w$  belongs to  $P_{15}, P_{16}, P_{25}P_{26}$ , or  $P_{36}$ . If  $w$  belongs to  $P_{15}, P_{16}, P_{36}$ ,  $A \cap V(P_{25} \cup P_{26})$ , or  $B \cap V(P_{26})$ , let  $R$  be defined as  $v_1P_{15}w, v_1P_{15}w, P_{24}, P_{24}$ , or  $P_{16}$ , respectively. Then  $H + Q - R$  is a hex with strictly more odd segments than  $H$ , contrary to the optimality of  $H$ .

So we may assume that  $w \in B \cap V(P_{25})$ . Note that this is the last case and is not symmetric with anything else, so it suffices to reduce to any of the previous cases. We now apply Lemma 3.2.1 to the paths  $v_1P_{14}u, uP_{14}v_4, Q$  and  $X := V(H) - (V(P_{14}) - \{w\})$  with  $A$  and  $B$  swapped. We may assume that the replacement paths do not change. If outcome (1) of the lemma holds, then there exist vertices  $y \in A \cap V(Q)$ ,  $z \in X$  and an  $H \cup Q$ -path  $R$  from  $y$  to  $z$ . If  $z \in B \cap V(P_{25})$ , we may assume it belongs to  $v_5P_{25}w$ . Then we can replace  $P_{25}$  by  $v_2P_{25}wQyRzP_{25}v_5$ ,  $Q$  by  $vQy$ , and

apply the case above where  $w \in A \cap V(P_{25})$ . If  $z \notin B$  or  $z$  is not on  $P_{25}$ , then we can replace  $Q$  with  $uQyRz$  and apply one of the previous cases.

So we may assume that the second outcome of the lemma holds. Thus there exist vertices  $a \in A \cap V(Q)$ ,  $b \in B \cap V(P_{14})$ ,  $c \in A \cap V(P_{14})$ , and  $d \in X$  and disjoint  $H \cup Q$ -paths  $R$  between  $a$  and  $b$  and  $S$  between  $c$  and  $d$ . We may assume that  $b$  belongs to  $v_1P_{14}u$ . Then we can replace  $P_{14}$  by  $v_1P_{14}bRaQuP_{14}v_4$  and  $Q$  by  $uP_{14}cRd$  which puts us in one of the previous cases unless  $d \in B \cap V(P_{25})$ . So we may assume  $d \in B \cap V(P_{25})$ , and that it belongs to  $wP_{25}v_5$ . Let  $H'$  be the hex obtained from  $H \cup S \cup R \cup Q$  by deleting  $V(P_{15} \cup v_5P_{25}d \cup P_{34} \cup P_{35} \cup P_{36}) - \{v_1, d, v_4, v_6\}$ . Then  $H'$  has nine odd segments, contrary to the optimality of  $H$ .  $\square$

**Lemma 3.3.3.** *Let  $G$  be an internally 4-connected bipartite graph. Then every optimal hex of  $G$  has at least six odd segments.*

*Proof.* Let  $(A, B)$  be a bipartition of  $G$  and let  $H$  be an optimal hex in  $G$  with feet and segments numbered as in the definition of a hex. By Lemma 3.3.2 we may assume  $H$  has exactly five odd segments and that that  $v_1, v_2, v_4 \in A$  and  $v_3, v_5, v_6 \in B$ . We now apply Lemma 3.2.1 to the paths  $P_{14}, P_{15}, P_{16}$  and set  $X := V(H) - V(P_{14} \cup P_{15} \cup P_{16})$ . We may assume that the replacement paths do not change.

Suppose first that outcome (2) of the lemma holds. Thus we may assume that there exist vertices  $u \in B \cap V(P_{14})$ ,  $w \in A \cap V(P_{15})$ ,  $y \in B \cap V(v_1P_{15}w)$  and  $z \in X$ , and disjoint  $H$ -paths  $Q$  from  $u$  to  $w$  and  $R$  from  $y$  to  $z$ . Then by symmetry we may assume that  $z$  belongs to one of  $P_{24}, P_{25}, P_{34}, P_{35}$ . Let  $T$  be, respectively,  $v_2P_{24}z \cup P_{25} \cup P_{26}$ ,  $v_5P_{25}z \cup P_{26}$ ,  $P_{25} \cup v_3P_{34}z$ ,  $P_{36} \cup v_5P_{35}z$ . Then the hexes  $H + (Q \cup R) - T$  have at least six odd segments which is a contradiction.

So we may assume that outcome (1) of the lemma holds. Thus there exists a vertex  $u \in B \cap V(P_{14})$  and an  $H$ -path  $Q$  from it to  $w \in X$ . Then by symmetry we may assume that  $w$  belongs to one of  $P_{24}, P_{25}, P_{34}$ , and  $P_{35}$ . For  $w$  on  $P_{24}, P_{34}, P_{35}$  or  $A \cap V(P_{25})$ , let  $R$  be, respectively,  $v_4P_{24}w, P_{36}, v_4P_{34}w, P_{34}$ . Then  $H + Q - R$  is a hex

with more odd segments than  $H$  which contradicts the optimality of  $H$ .

So we may assume  $w \in B \cap V(P_{25})$ . We now apply Lemma 3.2.1 to the paths  $v_1P_{14}u, uP_{14}v_4, Q$  and set  $X := V(H) - V(P_{14}) - \{w\}$ . Suppose that outcome (1) of the lemma holds. Thus there exist vertices  $y \in A \cap V(Q)$  and  $z \in X$  and an  $H \cup Q$ -path  $R$  from  $y$  to  $z$ . If  $z \in B \cap V(P_{25})$ , we may assume it belongs to  $v_5P_{25}w$ . Then we can replace  $P_{25}$  by  $v_2P_{25}wQyRzP_{25}v_5$ ,  $Q$  by  $uQy$ , and apply the case above where  $w \in A \cap V(P_{25})$ . If  $z \in B \cap V(P_{26})$ , then the hex  $H + (Q \cup R) - (v_5P_{25}w \cup v_2P_{26}z \cup P_{35})$  has nine odd segments which contradicts the optimality of  $H$ . If  $z \notin B$  or  $z$  is not on  $P_{25}$  or  $P_{26}$ , then we can replace  $Q$  with  $uQyRz$  and apply one of the previous cases.

So we may assume that the second outcome of the lemma holds. Thus there exist vertices  $a \in A \cap V(Q)$ ,  $b \in B \cap V(v_1P_{14}u)$ ,  $c \in A \cap V(P_{14})$  and  $d \in X$ , and disjoint  $H \cup Q$ -paths  $R$  between  $a$  and  $b$  and  $S$  between  $c$  and  $d$ . Then we can replace  $P_{14}$  by  $v_1P_{14}bRaQuP_{14}v_4$  and  $Q$  by  $uP_{14}cRd$  which puts us in one of the previous cases, unless  $d \in B$  and  $d$  is on  $P_{25}$  or  $P_{26}$ . Let  $F = S \cup Q \cup R$ . If  $d$  is on  $wP_{25}v_5$ , let  $J = P_{35} \cup P_{26} \cup v_5P_{25}d \cup P_{15}$ ; if  $d$  is on  $wP_{25}v_2$ , let  $J = P_{15} \cup P_{24} \cup P_{26} \cup v_2P_{25}d$ ; and if  $d$  is on  $P_{26}$ , let  $J = P_{24} \cup P_{35} \cup v_6P_{26}d$ . Then the hexes  $H + F - J$  have at least six odd segments, which contradicts the optimality of  $H$ .  $\square$

**Lemma 3.3.4.** *Let  $G$  be an internally 4-connected bipartite graph. Then in every optimal hex of  $G$  every segment is odd.*

*Proof.* Let  $(A, B)$  be a bipartition of  $G$  and let  $H$  be an optimal hex in  $G$  with feet and segments numbered as in the definition of a hex. By Lemma 3.3.3 we may assume  $H$  has exactly six odd segments and that that  $v_1, v_2, v_3, v_4 \in A$  and  $v_5, v_6 \in B$ . We now apply Lemma 3.2.1 to the paths  $P_{14}, P_{15}, P_{16}$  and set  $X := V(H) - V(P_{14} \cup P_{15} \cup P_{16})$ .

Suppose first that outcome (2) of the lemma holds. Thus there exist vertices  $u \in B \cap V(P_{14})$ ,  $w \in A \cap V(P_{15})$ ,  $y \in B$  and  $z \in X$ , and disjoint  $H$ -paths  $Q$  from  $u$  to  $w$  and  $R$  from  $y$  to  $z$ . By symmetry we may assume that  $z$  belongs to  $P_{24} \setminus v_2, P_{25}$ , or  $P_{26}$ . Let  $J$  be, respectively,  $P_{35} \cup zP_{24}v_2$ ,  $P_{34} \cup zP_{25}v_5$ , or  $P_{34} \cup zP_{26}v_6$ . Then the

hexes  $H + (Q \cup R) - J$  each have nine odd segments, which contradicts the optimality of  $H$ .

So we may assume that outcome (1) one of the lemma holds. Thus there exist  $u \in B \cap V(P_{14})$  and an  $H$ -path  $Q$  from it to  $w \in X$ . Then we may assume that  $w$  belongs to  $P_{24}$  or  $P_{25}$ . If  $w \in V(P_{24})$ , let  $R = v_4 P_{24} w$ . If  $w \in A \cap V(P_{25})$ , let  $R = P_{24}$ . Then  $H + Q - R$  is a hex with nine odd segments, a contradiction. Thus it remains to handle the case when  $w \in B \cap V(P_{25} \cup P_{26} \cup P_{35} \cup P_{36})$ .

We now forget  $w, Q, u$  and instead apply Lemma 3.2.2 to the paths  $P_{14}, P_{15}, P_{16}$  and set  $X := V(H) - V(P_{14} \cup P_{15} \cup P_{16})$ . Outcomes (A) and (C) give results already ruled out by the case analysis from applying Lemma 3.2.1, so we may assume that outcomes (B) or (D) hold.

Suppose that outcome (B) holds. Thus there exist vertices  $u \in B \cap V(P_{14})$ ,  $w \in B \cap X$ ,  $t \in A$  and  $s \in B$ , and a  $H$ -path  $Q = u \dots t \dots w$  and a  $H \cup Q$ -path  $R = t \dots s$ . By the previous analysis and symmetry we may assume that  $w \in V(P_{25})$ , so we are interested in where  $s$  lies. The above case analysis handles the cases where  $s$  is on  $P_{24}$  or  $P_{34}$ , so we need only worry about the case where  $s$  belongs to  $P_{15}$ ,  $w P_{25} v_5$ ,  $P_{35}$ ,  $P_{16}$ ,  $P_{26}$ ,  $P_{36}$ . Then, respectively, let  $J$  be defined as  $P_{24} \cup w P_{25} v_5$ ,  $P_{24} \cup w P_{25} s$ ,  $P_{24} \cup w P_{25} v_5$ ,  $P_{34} \cup P_{35} \cup P_{36}$ ,  $P_{34} \cup P_{35} \cup P_{36}$ ,  $P_{34} \cup P_{35} \cup v_3 P_{36} s$ . Then  $H + (Q \cup R) - J$  is a hex with nine odd segments, which contradicts the optimality of  $H$ .

So we must have outcome (D). Thus there exist vertices  $u, w \in V(P_{14}) \cap B$ ,  $r \in V(P_{14}) \cap A$ ,  $s \in A$  and  $x, y \in X \cap B$ , such that  $v_1, w, r, u, v_4$  occur on  $P_{14}$  in the order listed, and there exists a  $H$ -path  $Q = u \dots s \dots x$  and disjoint  $H \cup Q$ -paths  $R = w \dots s$  and  $S = r \dots y$ . Without loss of generality, we may assume that  $x \in V(P_{25})$ . By symmetry and taking advantage of the previous cases, we may assume that  $y$  belongs to  $x P_{25} v_5$ ,  $P_{26}$ ,  $P_{35}$ , or  $P_{36}$ . Let  $J$  be  $P_{15} \cup P_{34} \cup P_{35} \cup P_{36} \cup v_5 P_{25} y$ ,  $P_{24} \cup x P_{25} v_5 \cup P_{36}$ ,  $P_{15} \cup v_3 P_{35} y \cup P_{34} \cup P_{36}$ , or  $P_{24} \cup x P_{25} v_5 \cup v_3 P_{36} y$ , respectively. Then  $H + (Q \cup R \cup S) - J$  are each hexes with nine odd segments, a contradiction.  $\square$

*Proof of Theorem 3.1.3.* Let  $G$  be an internally 4-connected non-planar bipartite graph. By Theorem 3.1.2 the graph  $G$  has a hex, and hence it has an optimal hex  $H$ . By Lemma 3.3.4 every segment of  $H$  is odd, as desired.  $\square$

## CHAPTER IV

### PFAFFIAN ORIENTATIONS

#### 4.1 *Introduction*

As discussed in the introduction, in this chapter we investigate an alternative proof to a theorem of McCuaig, Robertson, Seymour, and Thomas [42, 31]:

**Theorem 4.1.1.** *A brace has a Pfaffian orientation if and only if it is isomorphic to the Heawood graph, or if it can be obtained from planar braces by repeated application of the trisum operation.*

We remind the reader of the definition of the trisum operation from Chapter 1. Specifically, let  $G_0$  be a graph and  $C$  a cycle of  $G_0$  of length 4 such that  $G_0 \setminus C$  contains a perfect matching. Let  $G_1, G_2, G_3$  be subgraphs of  $G_0$  such that  $G_1 \cup G_2 \cup G_3 = G_0$  and for distinct  $i, j \in \{1, 2, 3\}$ ,  $G_i \cap G_j = C$  and  $V(G_i) - V(C) \neq \emptyset$ . Let  $G$  be obtained from  $G_0$  by deleting some (possibly none) of the edges of  $C$ . Then  $G$  is a *trisum* of  $G_1, G_2$  and  $G_3$ .

The main theorem we require en route to this result is the following:

**Theorem 4.1.2.** *Let  $G$  be a nonplanar brace. Then  $G$  contains one of  $K_{3,3}$ , the Heawood graph, or Rotunda as a matching minor.*

In the previous chapter, we proved the following result:

**Theorem 4.1.3.** *Every internally 4-connected bipartite non-planar graph has an odd hex.*

Since it is not hard to see that braces are internally 4-connected, this provides a partial result in the direction of Theorem 4.1.2. Our goal in this chapter is to take

advantage of techniques similar to those used in the proof of Theorem 4.1.3 and apply them to provide a simpler proof of Theorem 4.1.2.

While Theorem 4.1.2 is of general interest in its own right, its most important aspect is its algorithmic implications. To that end, we state versions of the following two theorems from [42]:

**Theorem 4.1.4.** *Let  $G$  be a brace not isomorphic to the Heawood graph that contains the Heawood graph as a matching minor. Then  $G$  contains  $K_{3,3}$  as a matching minor.*

**Theorem 4.1.5.** *Let  $G$  be a brace that contains Rotunda as a matching minor. Then either  $G$  contains a set  $X$  of four vertices such that  $G \setminus X$  has three components or  $G$  contains  $K_{3,3}$  as a matching minor.*

We note that since  $K_{3,3}$  does not have a Pfaffian orientation, any graph that contains  $K_{3,3}$  as a matching minor does not have a Pfaffian orientation. These three theorems then allow us to sketch an algorithm, found in [42]. For a brace, check to see if it contains a set of 4 vertices whose deletion breaks the graph into 3 components which we will refer to as *trisectors*. If so, delete the graph on those vertices and recur in each component. Theorems found in [42] tell us the original graph has a Pfaffian orientation if and only if each component does. If there is no such separating set, check to see if the graph is isomorphic to the Heawood graph. Otherwise, if the graph is non-planar, by Theorem 4.1.2, it contains one of  $K_{3,3}$ , Heawood, or Rotunda as a matching minor, so by Theorems 4.1.4 and 4.1.5 contains  $K_{3,3}$  as a matching minor. Therefore, the graph does not have a Pfaffian orientation. If the graph is planar, then it does have a Pfaffian orientation and we can use an algorithm of Kasteleyn [22, 23] to find it. We will provide in section 4.7 an alternative algorithm for this problem that, in addition to avoiding the complex algorithms for finding trisectors also explicitly find the  $K_{3,3}$  matching minor in the case where the graph is not Pfaffian.

Since planarity testing can be done in linear time as can the problem of dividing a graph into braces, the limiting factor in the running-time of this algorithm is finding

the trisectors. In [42], the algorithm uses an  $O(n^2)$  procedure of Hopcroft and Tarjan [20], but more recent work alluded to in [47] by Hegde [19] allows us to improve this to  $O(n)$ . The overall running time of this algorithm is then  $O(n^2)$ .

In this chapter we will provide simpler proofs of all three of these core theorems, Theorems 4.1.2, 4.1.4, and 4.1.5 using similar techniques to those in Chapter 3 and then use these theorems to prove Theorem 4.1.1.

## 4.2 *Structural Lemmas*

While the main objective of this chapter is to prove Theorem 4.1.1, the goal of this section is to provide two of the core tools that we will require. We provide first a brief sketch of the techniques that we plan to develop and then develop the actual lemmas.

We begin with an odd subdivision of  $K_{3,3}$  with a maximal matching in the complement. We would like to choose these so that the number of vertices not included in the  $K_{3,3}$  and the matching is as small as possible. If we include everything, we're done. If some vertex is unmatched, we find three matching-augmenting paths from an unmatched vertex to the  $K_{3,3}$ . All these ends have the same parity; suppose they are  $B$ -vertices. We are interested in the  $B$ -ends of the subdivided paths of the  $K_{3,3}$  that these paths hit; specifically, we would like them to all be different. That this is always attainable if the underlying structure is an odd subdivision of  $K_{3,3}$  is the main result of the next section. If the underlying structure is not an odd subdivision of  $K_{3,3}$ , we can at least make two of the ends different, which is the other major result we require here.

Let us begin to formalize these notions.

First, we note that braces have a number of useful properties. The one most useful in this section is the following extension of Hall's Theorem, discussed in Chapter 1: Let  $G$  be a brace with bipartition  $(A, B)$ . Let  $X \subseteq A$  and  $N(X)$  be the neighbors of



$X$ . Then  $|N(X)| \geq |X| + 2$  or  $N(X) = B$ . We will refer to this as the *brace property* throughout this section.

**Definition.** Let  $G$  be a brace and  $H$  be a graph. Let  $K$  be a subgraph of  $G$  isomorphic to an odd subdivision of  $H$  with  $M$  a matching in  $G \setminus V(K)$ . Then we say that  $(K, M)$  is a *H-skein* and say that  $H$  is the *pattern* for  $(K, M)$  and  $K$  is the *base*. When  $H = K$ , we refer to  $(K, M)$  as simply a skein. We say that the *heft* of  $(K, M)$  is  $|V(K)| + |V(M)|$ . An *H-skein* is *maximal* if  $G$  contains no other *H-skein* with larger heft and is *perfect* if  $M$  is perfect in  $G \setminus V(K)$ .

Skeins contain a number of subdivided paths; we refer to a path in a graph  $G$  with both ends of degree at least 3 and every interior vertex of degree 2 as a *segment*. Vertices of degree at least 3 are referred to as *branch-vertices*. Further, let  $H$  be a subgraph of  $G$ . By an *H-path* in  $G$  we mean a path in  $G$  with at least one edge, both ends in  $V(H)$  and no other vertex or edge in  $H$ .

In this section, a skein will often be a  $K_{3,3}$ -skein which encourages the following two definitions:

**Definition.** Let  $G$  be a graph and  $H$  a subgraph of  $G$  isomorphic to a subdivision of  $K_{3,3}$ . Let  $v_1, v_2, \dots, v_6$  be the degree three vertices of  $H$  and  $P_1, P_2, \dots, P_9$  be the paths in  $H$  between  $v_i$ . We then refer to  $H$  as a *hex* or a *hex of G*, the  $v_i$  as the *feet* of  $H$ , and the  $P_i$  as the *segments* of  $H$ .

**Definition.** Let  $G$  be a graph and  $H$  a hex of  $G$ . We refer to a segment  $P$  of  $H$  as *even* if it has even length and *odd* otherwise. We refer to  $H$  as *odd* if all of its segments are odd. Further, we refer to a  $K_{3,3}$ -skein as a *weave*.

As mentioned in the brief description of the section, our main approach will be to take a skein and find an unmatched vertex along with  $M$ -alternating paths to the base. We give such a structure a name:

**Definition.** Let  $(K, M)$  be a skein in a bipartite graph  $G$  with bipartition  $(A, B)$ . Let  $S = V(G) \setminus (V(K) \cup V(M))$  and  $s \in S$ . Let  $P_1, P_2, P_3$  be  $M$ -alternating paths with ends  $s$  and, respectively,  $x, y, z \in V(K)$ , that are vertex disjoint except at  $s$  and disjoint from  $K$  except at their ends. Then we say that  $s, P_1, P_2, P_3$  form a *snarl* for  $(K, M)$  with ends  $x, y, z$  and root  $s$ . The *breadth* of a snarl is  $P_1 \cup P_2 \cup P_3$ .

We note that the root and ends of a snarl are of opposite parity. In the following proof we will discuss reversing a matching  $M$  along an  $M$ -alternating path  $P$  to find a new matching  $M'$ . By this we mean to remove from  $M$  every edge of  $P$  in  $M$  and simultaneously add to  $M$  every edge of  $P$  not in  $M$ . We show first that snarls exist:

**Lemma 4.2.1.** *Let  $G$  be a brace with bipartition  $(A, B)$ . Let  $(K, M)$  be a skein in  $G$  with  $M$  maximum in  $G \setminus V(K)$  and  $|V(K) \cap A| \geq 3$ . Let  $X = V(G) \setminus (V(K) \cup V(M))$  and  $x \in X \cap B$ . Let  $S$  be the set of vertices of  $B$  reachable by  $M$ -augmenting paths from  $x$ . Then  $S$  has at least three neighbors in  $K$ , and, for every three neighbors of  $S$  in  $K$ ,  $v_1, v_2, v_3$ , there exists a vertex  $s \in S$  and a matching  $M'$  such that  $(K, M')$  is a skein of the same heft as  $(K, M)$  and  $G$  contains a snarl in  $(K, M')$  with root  $s$  and ends  $v_1, v_2, v_3$ .*

*Proof.* We view  $S$  as an  $M$ -alternating tree rooted at  $x$ , by which we mean a tree such that every subpath of  $S$  with end  $x$  is  $M$ -alternating. Then  $S$  is a maximal  $M$ -alternating tree rooted at  $x$ . Since  $M$  is a maximum matching of  $G \setminus V(K)$  it follows that every leaf of  $S$  belongs to  $A \cap V(K)$ . We show first that  $S$  has at least three leaves. Suppose otherwise. Let  $S_A = S \cap A, S_B = S \cap B$  and  $k = |S_A \cap K|$ . Since each vertex of  $S_B$  except for  $x$  is matched to a vertex in  $S_A$  and every vertex of  $S_A \setminus V(K)$  is matched to a vertex of  $S_B$  under  $M$ , we have that  $|S_B| - 1 = |S_A| - k$ . Since  $S_A = N(S_B)$ , we have that  $|N(S_B)| = |S_B| + k - 1$ . By the brace property then, either  $k \geq 3$  or  $N(S_B)$  is all of  $A$ , in which case we still have that  $k \geq 3$  since  $|V(K) \cap A| \geq 3$ .

So there are three leaves of  $S$  in  $A \cap V(K)$ , say  $k_1, k_2, k_3$ . Let  $y$  be the unique vertex that belongs to all three subpaths of  $S$  from  $k_i$  to  $k_j$  for  $i, j \in \{1, 2, 3\}$ . Then the unique paths in  $S$  between  $y$  and  $k_1, k_2$  and  $k_3$  are vertex-disjoint. If we reverse  $M$  along the unique path between  $y$  and  $x$  in  $S$  to find a new matching  $M'$ , then these three paths are also  $M'$ -alternating, as desired.  $\square$

While snarls are useful tools, there are times when we need something that is less destructive to the matching. We provide the following lemma which is a generalization of (4.1) from [42]:

**Lemma 4.2.2.** *Let  $G$  be a brace with bipartition  $(A, B)$  and let  $(K, M)$  be a skein. Let  $X \subseteq V(K) \cap A$  be non-empty and let  $N = N(X) \cap V(K)$ . Suppose that  $|X| \geq |N| - 1$ . Let  $Y = B \setminus (N \cup V(M))$  and suppose  $Y$  is not empty. Then there exists an  $M$ -alternating path  $P$  in  $G$  with ends  $x$  and  $y$  with  $x \in X$ ,  $y \in Y$  and  $P$  vertex-disjoint from  $V(K)$  except at its ends.*

*Proof.* Let  $S$  be the set of all vertices reachable by  $M$ -alternating paths starting in  $X$  vertex-disjoint from  $V(K)$  except at its ends. Let  $S_A = S \cap A$  and  $S_B = S \cap B$ . Then the neighbors of  $S_A$  are exactly  $S_B$ , so we must have that either  $|S_B| \geq |S_A| + 2$  or that  $S_B = B$  since  $G$  is a brace. In the latter case, since  $Y$  is not empty, this gives the desired result. So we may assume that  $|S_B| \geq |S_A| + 2$ . Then  $|S_B \setminus N| \geq |S_A \setminus X| + 1$ . If every vertex in  $|S_B \setminus N|$  were matched, then we would have  $|S_A \setminus X| \geq |S_B \setminus N|$ , so some vertex in  $|S_B \setminus N|$  is unmatched which completes the proof.  $\square$

When we have an  $H$ -skein and a snarl with ends in  $A$  landing on subdivided edges of the base of the skein, we would like to discuss the  $A$ -ends of those edges. To facilitate this, we introduce the following two definitions:

**Definition.** Let  $v \in V(G)$  be a vertex of degree 2 with neighbors  $a, b$ . Then the process of *bicontracting*  $v$  is the deletion of  $v$  and the identification of  $a$  and  $b$ .

Note that bicontraction may create parallel edges or loops, though we only apply it in the context of bipartite graphs in which case loops do not result.

**Definition.** Let  $(K, M)$  be a skein. Let  $P$  be an odd segment of  $K$  with ends  $a \in A$ ,  $b \in B$ . Suppose  $(s, P_1, P_2, P_3)$  is a snarl with an end  $v \in P \cap B$ . Then we say that  $v$  *leans to*  $b$ . The *width* of a snarl is the number of different vertices its ends lean toward.

To illustrate this definition, we include in Figure 4.1 a snarl of width 2. We show in Theorem 4.2.5 that we may choose our snarls to have width at least 2, but begin with an illustrative lemma.

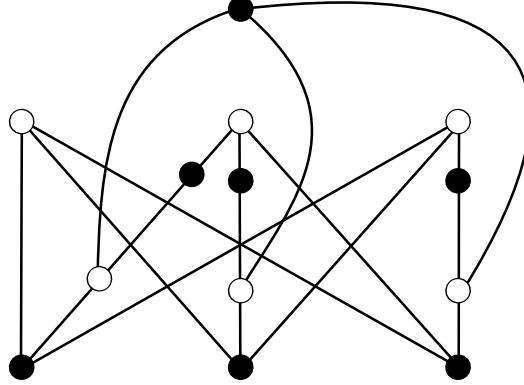


Figure 4.1: A snarl of width 2 in a weave

**Lemma 4.2.3.** *Let  $(K, M)$  be a maximal skein in a brace  $G$  with bipartition  $(A, B)$ . Let  $Q$  be a segment in  $K$  with ends  $u$  and  $v$ . Let  $(s, P_1, P_2, P_3)$  be a snarl for  $(K, M)$  with ends  $x, y, z$  with  $s \in B$ ,  $\{x, y, z\} \subseteq V(Q)$ . Let  $H = K \setminus (Q \setminus \{u, v\})$ . Then there exists a maximal skein  $(K', M')$  and a path  $Q'$  with ends  $u$  and  $v$  and each internal vertex of degree 2 in  $K'$  contained in  $K'$  such that  $K' = H \cup P'$ . Further, there exists a snarl  $(s', P'_1, P'_2, P'_3)$  for  $(K', M')$  with ends  $x', y', z'$  such that  $s' \in B$ ,  $x' \in V(Q')$ ,  $y' \in V(H) \setminus V(Q')$ , and  $z' \in V(K')$ .*

*Proof.* We proceed by means of contradiction. Suppose the outcome of the theorem is false. We may assume that, in order along  $Q$ , the vertices of the path read  $u, x, y, z, v$ .

Then choose the skein and snarl subject to the above such that  $|uQx| + |zQv|$  is as small as possible.

Let  $X = B \cap (V(P_1) \cup V(P_2) \cup V(P_3) \cup V(xQ_1z))$ . Let  $S = K \cup P_1 \cup P_2 \cup P_3$ . Note that in  $S$ , the number of neighbors of vertices of  $X$  is exactly  $|X|$ , so we may apply Lemma 4.2.2 to find an  $M$ -alternating path  $R$  from  $a \in X$  to  $b \notin N(X)$  unmatched under  $M$ .

Suppose  $b \notin V(Q)$ . If  $a \in yQx$ , then take  $s' = a, x' = x, y' = y, z' = b, P'_1 = uQx, P'_2 = uQy, P'_3 = R, Q' = vQyP_2sP_1xQa$  and everything else stays the same. Then this either increases the heft of our skein if  $w$  is an unmatched vertex not in  $K$  (because  $s$  is now counted in the heft and we can modify  $M$  to match  $a$  along this new path) or fulfills the conditions of the theorem if  $b \in V(H)$ . Note that it is impossible to find a skein of larger heft by the maximality of  $(K, M)$ . The argument for  $a \in yQz$  is nearly identical, so we omit it. If  $a \in P_1$ , take  $s' = a, x' = x, y' = y, z' = b, P'_1 = aP_1x, P'_2 = aP_1sP_2y, P'_3 = R$  and the rest as before. Note that a similar argument works for  $a \in P_2$  or  $a \in P_3$ .

So we may assume that  $b \in V(Q_1)$ . We argue for  $b \in V(xQu)$  but note that similar reasoning works for  $b \in V(vQz)$ . So assume  $b \in V(xQu)$ . If  $a \in V(P_1)$ , take  $x' = b, P'_1 = sP_1aRb$  and everything else the same to decrease  $|uQx'| + |zQv|$  which is a contradiction. If  $a \in V(P_2)$ , take  $s' = a, x' = b, y' = y, z' = z, P'_1 = R, P'_2 = aP_2y, P'_3 = aP_2sP_3z$  and everything else the same to again decrease  $|uQ_1x'| + |zQ_1v|$ . Note that  $a \in V(P_3)$  is nearly identical. For  $a \in V(yQx)$ , take  $s' = s, x' = b, y' = y, z' = z, Q' = vQaRbQu, P'_1 = sP_1xQb, P'_2 = P_2, P'_3 = P_3$ . Finally, if  $a \in V(zQy)$ , take  $s' = s, x' = b, y' = a, z' = z, Q' = vQaRbQu, P'_1 = sP_1xQ_1b, P'_2 = sP_2yQ_1a, P'_3 = P_3$ . In both of these cases, we reduce  $|uQ_1x'| + |zQ_1v|$ , which is a contradiction and completes the proof.

□

Let  $H$  be a 3-regular graph and let  $(K, M)$  be an  $H$ -skein in a bipartite graph

with bipartition  $(A, B)$ . Let  $v \in V(K) \cap B$  be a branch-vertex of  $K$ . Let  $Q$  be the union of the segments of  $K$  incident with  $v$  and let  $\sigma$  be a snarl for  $(K, M)$ . By a  $v$ -span of  $\sigma$  we mean the smallest connected subgraph of  $Q$  that includes all the ends of  $\sigma$  that belong to  $Q$ . The interior of a  $v$ -span is the set of all vertices that have degree at least two in the  $v$ -span.

In what follows we will often have a skein  $(K, M)$  perhaps additionally with some number of snarls the union of whose breadths is  $S$ . Let  $H = K \cup S$ . Let  $P$  be a union of  $H$ -paths and  $Q$  be a subgraph of  $H$ . Then by  $H + P - Q$  we mean the graph obtained from  $H \cup P$  by deleting all edges of  $Q$  and then deleting all resulting isolated vertices. In the cases we consider, this resulting graph will be skein of some fixed pattern or the union of a skein and a snarl; in the latter case we additionally specify the root of the snarl, from which the skein and snarl may be uniquely determined. We will refer to this as the root of  $H + P - Q$ . A typical application will be when  $P$  and  $Q$  are path, but we will need more complicated choices. In these applications, we may need to alter  $M$  by an application of Lemma 4.2.1 and additionally we will grow the matching  $M$  maximally in the complement of the resulting hex; in each case there will be a natural way to do this since  $P$  and  $Q$  will typically be the unions of  $M$ -alternating paths. For the convenience of the reader, when the resulting skein is a weave, we will additionally specify the feet of the hexes that we find and refer to them as the feet of  $H + P - Q$ .

**Lemma 4.2.4.** *Let  $H$  be a bipartite 3-regular graph and let  $(K, M)$  be a maximal  $H$ -skein in a brace  $G$  with bipartition  $(A, B)$ . If  $M$  is not a perfect matching of  $G \setminus V(K)$  then there exists a maximal  $H$ -skein  $(K', M')$  of the same heft as  $(K, M)$  and a snarl  $\sigma = (s, P_1, P_2, P_3)$  for  $(K', M')$  such that  $s \in B$  and either*

- (1)  $\sigma$  has width three or
- (2)  $\sigma$  has width two and there exist a branch-vertex  $v \in V(K') \cap A$  of  $K'$  and an

*$M'$ -alternating  $K'$ -path  $R$  with ends  $u$  and  $w$  such that  $u$  belongs to the interior of the  $v$ -span of  $\sigma$ ,  $w$  does not belong to any of the segments of  $K'$  incident with  $v$  and  $R$  is disjoint from each of the paths of  $\sigma$*

*Proof.* Since  $M$  is not perfect and  $G$  and  $K$  both contain perfect matchings (since  $G$  is a brace and  $K$  is an odd subdivision of a bipartite three-regular graph) there exists a vertex  $s \in B \setminus V(K)$  that is unmatched by  $M$ . Let  $S$  be the set of all vertices of  $B$  that can be reached from  $s$  by an  $M$ -alternating path.

Let  $x, y \in V(K)$  be the neighbors of vertices in  $S$ . Then we say that a path  $T$  with ends  $x$  and  $y$  is an  $S$ -arch if there exists a vertex  $t \in S$  and two  $M$ -alternating paths  $T_1, T_2$  with ends  $t$  and  $x, y$  respectively such that  $T = T_1 \cup T_2$ . Note that for any two neighbors of  $S$ , there is an  $S$ -arch,  $T$ , provided by Lemma 4.2.1 (up to an alteration of  $M$ ) and that the modified matching saturates every vertex of  $S \cup N(S) - V(K) - T$ .

Let  $A_0$  be the set of all branch-vertices of  $K$  that belong to  $A$ . Let  $v \in A_0$  and let  $Q$  be the union of the segments of  $K$  incident with  $v$ . By a  $v$ -span of  $S$  we mean the smallest connected subgraph of  $Q$  that includes all the neighbors of vertices in  $S$  that belong to  $Q$ . Let  $J$  denote the  $v$ -span. By a  $v$ -antispans we mean the unique subgraph  $L$  of  $Q$  such that  $L$  has no isolated vertices,  $L \cup J = Q$ , and  $E(J) \cap E(L)$  is empty.

Choose  $K, M, s$  such that

- (i) the number of vertices in  $A_0$  with non-null  $v$ -span is maximum, and, subject to (i),
- (ii) the union of all  $v$ -antispanns over all  $v \in A_0$  is minimum.

Let  $A_1$  be the set of all vertices  $v \in A_0$  such that the  $v$ -span is non-null. Then  $A_1$  is not empty by Lemma 4.2.1. We may assume that  $|A_1| \leq 2$ , since otherwise the lemma holds by Lemma 4.2.1. Let  $X$  be the union of  $S$  and all vertices that

belong to  $B \cap V(J)$  for some  $v$ -span  $J$ , where  $v \in A_1$ . Since  $|X| \leq |N(X)| + 1$ , by Lemma 4.2.2 we can find an  $M$ -alternating path  $R$  with an end  $u \in X$  and another end  $w$  in  $V(G) \setminus (V(M) \cup N(X))$ . Suppose  $u \in S$ . Then  $w \notin V(G) \setminus V(K)$  since we have an  $M$ -alternating path between  $s$  and  $w$  which would contradict the maximality of  $M$ . If  $w \in V(K)$ , then we improve either (i) or (ii) immediately. It follows that  $u \in V(X) \setminus S$ .

Then  $u$  lies in the  $v$ -span  $J$  of some  $v \in A_1$ . Let the segments incident with  $v$  be  $Q_1, Q_2, Q_3$ , their union be  $Q_0$ , and their ends be, respectively  $a, b, c$ . Without loss of generality, we may assume  $u \in J \cap Q_1$ . Let  $x$  be the leaf of  $J$  on  $uQ_1a$ .

By the symmetry between  $Q_2$  and  $Q_3$ , we may assume  $w \notin Q_3 \setminus \{v\}$ . Let  $y$  be another leaf of  $J$  chosen such that  $y$  is not on  $Q_2$  if possible. Let  $T$  be an  $S$ -arch with ends  $x$  and  $y$  and let  $M_1$  be a matching in  $G \setminus V(K \cup T)$ , as provided by Lemma 4.2.1. Let  $K' = K + T - xQ_1y$  if  $y \in V(Q_1)$  and  $K' = K + T - vQ_1x$  if  $y \notin V(Q_1)$ . Then by using edges of  $xQ_1y$  or  $vQ_1x$ , the matching  $M_1$  can be enlarged to a matching  $M'$  in  $G \setminus (V(K') \cup \{u\})$  such that  $(K', M')$  is a skein of the same heft as  $(K, M)$ .

If  $|A_1| = 1$ , then the triple  $K', M', u$  contradicts the choice  $K, M, s$ . Indeed  $w \in V(K)$  by the maximality of  $M$ ; if  $w \notin V(Q_0)$ , then (i) is violated and otherwise (ii) is violated given that  $w \notin V(Q_3) \setminus \{v\}$ , and, if  $y \in V(Q_2)$ , then  $Q_3$  is a subgraph of the  $v$ -antispans of  $S$  by the choice of  $y$ .

Thus  $|A_1| = 2$ . If  $w \notin V(K)$ , then  $(K', M')$  has larger heft than  $(K, M)$ . If  $w \notin V(Q_0)$ , the path  $R$  satisfies (2) of the lemma and we can find an appropriate snarl in  $S$  by Lemma 4.2.1.

Thus we may assume  $w \in V(Q_0)$ . Now we redefine  $K'$  as  $K' = K + R - uQ_1w$  if  $w \in V(Q_1)$  and  $K' = K + R - uQ_1v$  otherwise (that is,  $w \in V(Q_2) \setminus \{v\}$ ), and we let  $M'$  be the matching obtained from  $M \setminus E(R)$  by adding a perfect matching of  $uQ_1w \setminus \{u, w\}$  or  $uQ_1v \setminus \{u, v\}$  respectively. Then  $(K', M')$  is an  $H$ -skein of the same heft as  $(K, M)$ , and the triple  $K', M', s$  contradicts the choice of  $K, M, s$  unless  $w$



and  $y$  both belong to  $Q_2 \setminus \{v\}$ .

Thus we may assume  $w, y \in V(Q_2) \setminus \{v\}$  and hence  $Q_3$  is a subgraph of the  $v$ -antispan. We now choose  $K, M, s, u, w, R$  subject to the above such that

(iii) the length of  $wQ_2b$  is minimum

Let  $X' = X \cup ((yQ_2w \cup V(R) \cap B)$ . Then  $|N(X')| = |X'| + 1$ , so by Lemma 4.2.2 we can find an  $M$ -alternating path,  $P$ , with one end  $p \in X'$  and the other end  $q \in V(G) \setminus (V(M) \cup N(X'))$ . Let  $v^*$  be the vertex of  $A_1$  other than  $v$  and  $J^*$  its span. As before, we cannot have  $p \in S$  and by the preceding arguments and (iii) we cannot have  $p \in V(R)$ , so we may assume  $p \in V(J \cup J^*)$ . If  $p \in V(J^*)$ , then by the preceding arguments the path  $P$  is positioned in the same way relative to  $J^*$  as the path  $R$  is positioned relative to  $J$ , thus making  $J$  and  $J^*$  symmetric. Let  $y'$  and  $Q'_2$  be the symmetric images of  $y$  and  $Q_2$  in  $J^*$ . Let  $X'' = X' \cup ((y'Q'_2q \cup V(P) \cap B)$ . But then  $|N(X'')| = |X'| + 1$ , so by another application of Lemma 4.2.2, we find yet another path  $P'$  with ends  $p' \in X''$  and  $q' \in V(G) \setminus (V(M) \cup N(X''))$ , and, from the newly gained symmetry between  $J$  and  $J^*$  we may assume that  $p' \in J$ . But now the path  $P'$  can play the role of  $P$  and so we may assume that  $p \in V(J)$ .

Suppose that  $p \in vQ_2y$ . Then by the above arguments, we must have  $q \in xQ_1a$ . Now we redefine  $K'$  as  $K' = K + (R \cup P) - (uQ_1q \cup pQ_2w)$ , and we let  $M'$  be the matching obtained from  $M \setminus E(R \cup P)$  by adding a perfect matching of  $uQ_1q \cup pQ_2w \setminus \{u, w, p, q\}$ . Then  $(K', M')$  is an  $H$ -skein of the same heft as  $(K, M)$ , and the triple  $K', M', s$  violates (ii).

So  $p \in yQ_2w$ . Let  $T$  be an  $S$ -arch with ends  $x$  and  $y$  and let  $M_1$  be a matching in  $G \setminus V(K \cup T)$ , as provided by Lemma 4.2.1. Then  $q$  lies in one of  $V(G \setminus K), V(K \setminus Q_0), Q_1, Q_2, Q_3$ . Let  $C$  be, respectively,  $uQ_1x \cup yQ_2w, uQ_1v, vQ_2p \cup uQ_1q, pQ_2w, qQ_3vQ_2p \cup uQ_1v$ . Let  $D$  be, respectively,  $T \cup R, R, P \cup R, P, P \cup R$ . Let  $X_C$  be the vertices of degree one in  $C$  and  $M_C$  be a perfect matching in  $C \setminus X_C$ . Let  $M'$  be the matching obtained from  $M \setminus E(D)$  by adding  $M_C$  if  $q \in V(K)$ . If  $q \in V(G \setminus K)$ , let  $M'$  be the

matching obtained from  $M \setminus E(T \cup R)$  by adding  $M_C$  and a perfect matching of  $P$ . Then let  $K' = K + D - C$ . Note that  $(K', M')$  has at least the same heft as  $(K, M)$ .

Note that if  $q \in V(G \setminus K)$  the skein  $(K', M')$  has larger heft which contradicts the maximality of  $(K, M)$ . If  $q \in V(K \setminus Q_0)$ , then a snarl found in  $S$  from the skein  $(K', M')$  by Lemma 4.2.1 satisfies outcome (2) with  $P$  serving as  $R$ . If  $q \in V(Q_1 \cup Q_3)$ , then the triple  $K', M', w$  contradicts the choice of  $K, M, s$  since it contradicts (ii) above. If  $q \in V(Q_2)$ , then the tuple  $K', M', s, u, q, R \cup wQ_2q$  contradicts the choice of  $K, M, s, u, w, R$  since it contradicts (iii).

□

A immediate corollary of this is often useful:

**Theorem 4.2.5.** *Let  $(K, M)$  be a maximal skein in a brace  $G$  with bipartition  $(A, B)$  whose pattern is one of Rotunda,  $K_{3,3}$  or the Heawood graph. Then if  $V(K) \cup V(M) \neq V(G)$ ,  $G$  contains a maximal weave  $(K', M')$  and a snarl for  $(K', M')$ ,  $(s, P_1, P_2, P_3)$ , of width 2.*

Much of the argument for the preceding theorem concerned the behavior of snarls near vertices of degree three. To better understand this behavior we present a preliminary definition and then the following lemmas.

**Definition.** Let  $G$  be a bipartite graph with bipartition  $(A, B)$ . Let  $K$  be a subgraph of  $G$  such that no block is a cycle. Let  $u_1, u_2 \in V(K)$  be branch-vertices of the same parity, say  $A$  and let  $P$  be a segment of  $K$  with ends  $u_1$  and  $u_2$ . Let  $Q_1, \dots, Q_k$  be the segments incident with  $u_1$  and  $u_2$  other than  $P$  and let  $Q = \bigcup_{i=1}^k Q_i$ . Let  $M$  be a matching in the complement of  $K \cup Q$ . Then an *augmenting sequence* subject to  $M, K$ , and  $P$  is a sequence of  $M$ -alternating  $Q \cup K$ -paths  $R_1, \dots, R_m$  disjoint from each other with ends  $v_1, v_2, \dots, v_{2m}$  such that the ends of  $R_i$  are  $v_{2i-1}$  and  $v_{2i}$  and  $v_1 \in B \cap V(P)$ . Further, for every  $i \in \{1, \dots, m-1\}$ , there exists  $j \in \{1, \dots, k\}$  and  $h \in \{1, 2\}$  such that  $u_h$  is incident with  $Q_j$ ,  $v_{2i} \in V(Q_j)$  and  $v_{2i+1} \in u_h Q_j v_{2i}$  and

if  $v_{2i+1}$  and  $v_{2i'+1}$  belong to  $Q_j$  with  $i < i'$  then  $v_{2i+1} \in u_h P_{v_{2i+1}}$ . We refer to each  $R_i$  as an *augmentation* and the *length* of an augmenting sequence is the number of augmentations it contains.

We show first that augmenting sequences exist.

**Lemma 4.2.6.** *Let  $G$  be a bipartite graph with bipartition  $(A, B)$ . Let  $K$  be a subgraph of  $G$  such that no block is a cycle. Let  $u_1, u_2 \in V(K)$  be branch-vertices of the same parity, say  $A$  and let  $P$  be a segment of  $K$  with ends  $u_1$  and  $u_2$ . Let  $Q_1, \dots, Q_k$  be the segments incident with  $u_1$  and  $u_2$  other than  $P$  and let  $Q = \bigcup_{i=1}^k Q_i$ . Let  $M$  be a matching in the complement of  $K \cup Q$ . Then there exists an augmenting sequence subject to  $M$ ,  $K$ , and  $P$ ,  $R_1, \dots, R_m$  such that the ends of  $R_i$ ,  $1 \leq i \leq m$  are  $v_{2i-1}$  and  $v_{2i}$ . Further,  $v_{2m} \in V(G) \setminus V(Q \cup P \cup M)$ .*

*Proof.* We define the span of an augmenting sequence as follows. Let  $R = R_1, \dots, R_m$  be an augmenting sequence subject to  $M, K$  and  $P$  so that the endpoints of  $R_i$  are  $v_{2i-1}$  and  $v_{2i}$  and so that, for  $i \in \{1, 2, \dots, 2m\}$ ,  $v_i \in Q \cup P$ . We may assume that each of the  $v_i \in V(Q)$  since otherwise the lemma holds. Let  $T$  be the smallest tree contained in  $Q \cup P$  that contains  $P$  and each of the  $v_i$ ,  $i \in \{1, 2, \dots, 2m\}$ . Then the *span* of  $R$  is  $V(T)$ . Let  $R'$  be an  $M$ -alternating  $P \cup Q$  path with one end in  $V(T) \cap B$ , disjoint from  $R$ . Then it is clear that there exists an augmenting sequence subject to  $M$  and  $P$  that contains  $R'$  and whose paths are a subset of the paths of  $R$  along with  $R'$ .

Let  $S$  be the union of the spans of all augmenting sequences subject to  $P$  and  $M$  and  $R$  be the union of all the paths in those augmenting sequences. Let  $X = V(S \cup R) \cap B$ . Then  $|N(X)| = |X| + 1$ , so by Lemma 4.2.2, there exists an  $M$ -alternating path,  $R'$ , with one end,  $v \in V(S \cup R) \cap B$  and the other,  $v'$  in  $V(G) \setminus V(S \cup R \cup M)$ . Since there is an augmenting sequence subject to  $P$  and  $M$  containing  $R'$  (either by the above argument or by rerouting one of the previous paths along  $R'$ ),  $v'$  must be

in the span of all augmenting sequences subject to  $P$  and  $M$  which is a contradiction.

□

We now make use of augmenting sequences to better understand the neighborhood of branch vertices of degree 3.

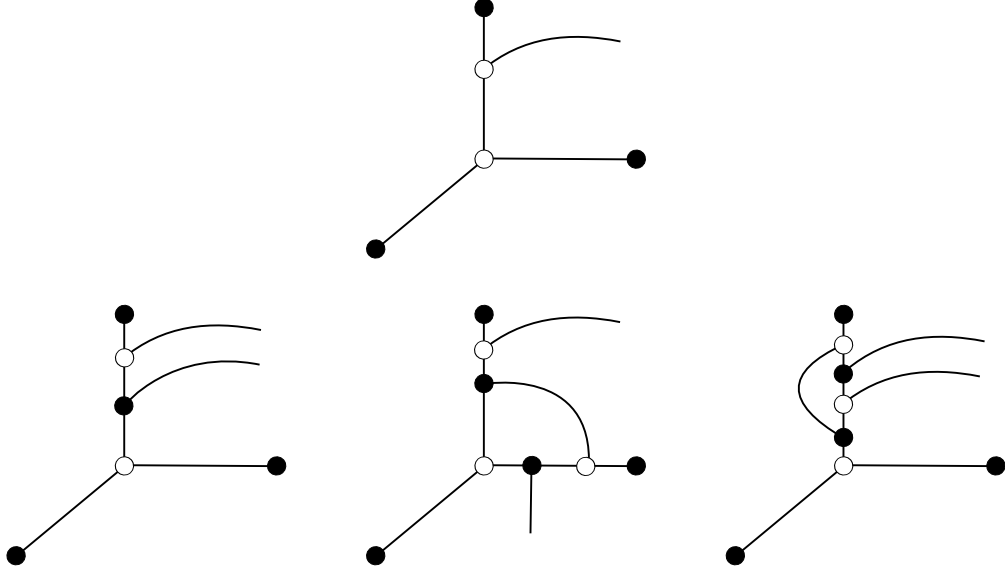


Figure 4.2: The outcomes of Lemma 4.2.7

**Lemma 4.2.7.** *Let  $G$  be a brace with bipartition  $(A, B)$ , let  $H$  be a graph with no vertices of degree two, let  $(K, M)$  be a maximal  $H$ -skein in  $G$ , let  $v \in B$  be a branch-vertex of  $K$  of degree three, let  $P_1, P_2, P_3$  be the three segments of  $K$  incident with  $v$ , and let  $v_1, v_2, v_3$ , respectively, be their other ends. Let  $q \in B \cap V(P_1 \cup P_2)$ , let  $p \in V(K) \cap A - V(P_1 \cup P_2 \cup P_3)$ , and let  $R$  be an  $M$ -alternating  $K$ -path in  $G$  with ends  $p$  and  $q$ . Then there exists a maximal  $H$ -skein  $(K', M')$ , where  $K'$  is obtained from  $K$  by replacing the paths  $P_1, P_2, P_3$  by paths  $P'_1, P'_2, P'_3$ , respectively, and  $P'_i$  has ends  $v'$  and  $v_i$ . Furthermore, there exist an integer  $i$  in  $\{1, 2\}$ , vertices  $q' \in V(P_i) \cap B$  and  $p' \in V(K') \cap A - V(P'_1 \cup P'_2 \cup P'_3)$ , an  $M$ -alternating  $K'$ -path  $R'$  with ends  $p'$  and  $q'$  and either*

- (i) vertices  $u_1 \in vP'_i q' \cap A$ ,  $u_2 \in V(K') \cap B - V(P'_1 \cup P'_2 \cup P'_3)$  and an  $M'$ -alternating  $(K' \cup R')$ -path with ends  $u_1$  and  $u_2$ , or
- (ii) vertices  $u_1 \in vP'_i q' \cap A$ ,  $u_2 \in P_{3-i} \cap B$ ,  $u_3 \in u_2 P_{3-i} v \cap A$  and  $u_4 \in V(K') \cap B - V(P'_1 \cup P'_2 \cup P'_3)$  and two disjoint  $M'$ -alternating  $(K' \cup R')$ -paths, one with ends  $u_1$  and  $u_2$ , and the other with ends  $u_3$  and  $u_4$ , or
- (iii) vertices  $u_1 \in vP'_i q' \cap A$ ,  $u_2 \in q' P'_i v'_i \cap B$ ,  $u_3 \in u_2 P'_i q' \cap A$  and  $u_4 \in V(K') \cap B - V(P'_1 \cup P'_2 \cup P'_3)$  and two disjoint  $M'$ -alternating  $(K' \cup R')$ -paths, one with ends  $u_1$  and  $u_2$ , and the other with ends  $u_3$  and  $u_4$ .

*Proof.* Let  $X = V(K) \cap B - V(P_1 \cup P_2 \cup P_3)$ . Let  $P$  be the element of  $\{P_1, P_2, P_3\}$  containing  $q$ . By Lemma 4.2.6, there exists an augmenting sequence  $R_1, \dots, R_n$  subject to  $P$  and  $M$  such that the ends of  $R_i$  are  $u_{2i-1}$  and  $u_{2i}$  and  $u_{2n} \in V(K) \cap B - V(P_1 \cup P_2 \cup P_3)$ . Choose  $v, P_1, P_2, P_3, M$  and such an augmenting sequence so that the augmenting sequence has as few augmentations as possible and, subject to that, such that as few of the ends of the augmentations lie on  $P_3$  as possible. Note that only  $u_1 \in V(vPq)$  since otherwise we immediately have a shorter augmenting sequence. Therefore, if the sequence has length 1 or 2 and avoids  $R$  and  $P_3$ , we are done, so we may assume it has length at least 3 or includes a vertex of  $R$  or  $P_3$ . Note that if it has length 1, this doesn't occur, so we may assume the length of the augmenting sequence is at least 2.

We may assume without loss of generality that  $q$  lies on  $P_1$ .

Suppose first that  $u_2 \in V(R)$ . Then  $u_3 \in V(qRu_2)$ . If  $u_4 \in X$ , then we take  $P'_1 = vP_1 u_1 R_1 u_2 R q P_1 v_1$ ,  $R'_1 = u_1 R_1 q$ ,  $u'_2 = q$ ,  $q' = u_2$ ,  $R' = pRu_2$  to find an appropriate augmenting sequence as desired. Suppose instead that  $u_4 \in V(P_2)$ . Then we take  $q' = u_4$ ,  $R' = pRu_3 R_2 u_4$  to find a shorter augmenting sequence having exchanged  $v_1$  and  $v_2$ . If  $u_4 \in V(P_3)$ , take  $v' = u_4$ ,  $P'_1 = u_4 R_2 u_3 R q P_1 v_1$ ,  $P'_2 = u_4 P_3 v P_2 v_2$ ,  $P'_3 = u_4 P_3 v_3$ ,  $q' = v$ ,  $R' = pRu_2 R_1 u_1 P_1 v$ ,  $u'_1 = u_5$  to find a shorter augmenting sequence.

Finally, if  $u_4 \in V(P_1)$ , take  $q' = u_4, R' = pRu_3R_2u_4$  to find a shorter augmenting sequence.

Suppose that  $u_2 \in V(P_3)$ . Then  $u_3 \in V(vP_2u_2)$ . Then take  $v' = u_2, P'_1 = u_2R_1u_1P_1v_1, P'_2 = u_2P_3vP_2v_2, P'_3 = u_2P_3v_3, u'_2 = v, R'_1 = u_1P_1v$  to find an augmenting sequence of the same length with fewer ends on  $P_3$ .

So we may assume the length of the augmenting sequence is at least 3.

Suppose that  $u_2 \in V(qP_1v_1), u_3 \in V(qP_1u_3)$ . Suppose  $u_4 \in V(R)$ . Then take  $P'_1 = vP_1u_1R_1u_2P_1v_1, q' = u_2, R' = pRu_4R_2u_3P_1u_2, R'_1 = u_1P_1qRu_5R_3$  to find a shorter augmenting sequence. If  $u_4 \in V(qP_1v)$ , then replacing  $P_1$  by  $vP_1u_3R_2u_4P_1v_1$  and  $R_1$  by  $u_1R_1u_2P_1u_5R_3$  reduces the length of the augmenting sequence. If  $u_4 \in V(P_3)$ , then take  $v' = u_4, u'_1 = u_3, u'_2 = u_2, u'_3 = u_1, u'_4 = v, P'_1 = u_4R_2u_3P_1u_1R_1u_2P_1v_1, P'_2 = u_4P_3vP_2v_2, P'_3 = u_4P_3v_3, R'_1 = R_2, R'_2 = u_1P_1v$  to find the same augmenting sequence, but with  $u_4 \in V(P_2)$ . So we may assume  $u_4 \in V(P_2)$ . Then take  $q' = u_4, P'_1 = vP_1u_1R_1u_2P_1v_1, R' = pRqP_1u_3R_2u_4$  to find a shorter augmenting sequence having exchanged  $v_1$  and  $v_2$ .

So we may assume that  $u_2 \in V(P_2)$ . Note that if  $u_2$  were in  $V(P_3)$ , taking  $v' = u_2, P'_1 = u_2R_1u_1P_1v_1, P'_2 = u_2P_3vP_2v_2, P'_3 = u_2P_3v_3$  to find the same situation except with  $u_2 \in V(P_2)$ . If  $u_4 \in V(qP_1v)$ , then take  $v' = u_4, u'_1 = u_5, P'_1 = u_4P_1v_1, P'_2 = u_4P_1u_1R_1u_2v_2, P'_3 = u_4R_2u_3P_2vP_3v_3$  which gives a shorter augmenting sequence having exchanged  $v_1$  and  $v_2$ . Suppose  $u_4 \in V(R)$ . Then take  $v' = v, P'_1 = vP_2u_3R_2u_4RqP_1v_1, P'_2 = vP_1u_1R_1u_2P_2v_2, P'_3 = P_3, q' = u_4, R' = pRu_4, u'_1 = u_3, u'_2 = u_2, u'_3 = u_1, u'_4 = q, R'_1 = u_2P_2u_3, R'_2 = u_1P_1q$  which gives the previous case. If  $u_4 \in V(P_2)$ , we can replace  $P_2$  by  $vP_2u_3R_2u_4P_2v_2$  and  $R_1$  by  $u_1R_1u_2P_2u_5R_3$  to reduce the length of the augmenting sequence. So we may assume  $u_4 \in V(P_3)$ . But then take  $v' = u_2, P'_1 = v_1P_1u_1R_1u_2, P'_2 = u_2P_2v_2, P'_3 = v_3P_3u_4R_2u_3P_2u_2, R'_1 = u_1P_1vP_3u_5R_3$  which gives a shorter augmenting sequence.

□

### 4.3 Weaves

In the case of skeins patterned after Rotunda and the Heawood graph, Theorem 4.2.5 suffices, but for weaves, we require width 3 snarls.

Before we begin, we provide a lemma that shows an application of bicontraction.

**Lemma 4.3.1.** *Let  $G$  be a graph,  $w$  a vertex of  $G$  of degree two, and let  $G'$  be obtained from  $G$  by bicontracting  $w$ . Let  $H$  be a three-regular graph. If  $G'$  has an  $H$ -skein of heft  $k$ , then  $G$  contains an  $H$ -skein of heft  $k + 2$  and if  $G'$  contains an  $H$ -skein of heft  $k$  and a snarl for it of width  $w$ , then  $G$  also has an  $H$ -skein of heft  $k + 2$  and a snarl for it of width  $w$ .*

*Proof.* Let the neighbors of  $w$  be  $u$  and  $v$  and let  $x$  be the vertex of  $G'$  obtained by identifying  $u$  and  $v$ . Suppose  $G'$  has a weave  $(K, M)$ . We present the following argument assuming that  $G'$  additionally contains a snarl  $\sigma = (s, P_1, P_2, P_3)$  with respect to  $(K, M)$  but note that if  $G$  does not, then the same argument proves the desired conclusion.

If  $x \in V(M \setminus \sigma)$ , then let  $y$  be its neighbor under  $M$ . Then  $y$  is adjacent to either  $u$  or  $v$ , say  $u$ , in which case  $(K, M \setminus \{xy\} \cup \{uy, wv\})$  along with  $\sigma$  is as desired.

If  $x = s$ , let the three neighbors of  $x$  along  $P_1, P_2, P_3$  be, respectively,  $p_1, p_2, p_3$ . Then one of  $u, v$  is adjacent to two of  $\{p_1, p_2, p_3\}$ , say  $u$  adjacent to  $p_1$  and  $p_2$ . If  $u$  is adjacent to  $p_3$ , let  $P'_3 = P_3$  and otherwise let  $P'_3 = uwvP_3$ . Then let  $M' = M \cup \{wv\}$  and the weave  $(K, M')$  and snarl  $(u, P_1, P_2, P'_3)$  is as desired.

If  $x \in V(P_1 \cup P_2 \cup P_3) \setminus V(K)$ , we may assume  $x \in V(P_1)$  by symmetry and that  $x$  and  $p_1$  are matched under  $M$ . Then let  $P_1 = s \dots p_1 x p_2 \dots$ . We may assume that  $u$  is adjacent to  $p_1$ . If  $u$  is also adjacent to  $p_2$ , let  $P'_1 = s P_1 p_1 u p_2 P_1$  and  $P'_1 = s P_1 p_1 u w v p_2 P_1$  otherwise. Let  $M' = M \setminus \{p_1 x\} \cup \{p_1 u, wv\}$ , in which case the weave  $(K, M')$  with snarl  $(s, P'_1, P_2, P_3)$  is as desired.

So we have that  $x \in V(K)$ . If  $x$  is not a branch vertex of  $K$ , then let its neighbors

in  $K$  be  $q_1, q_2$  on a segment  $Q$  of  $K$  where  $Q = a...q_1xq_2...b$  where  $a$  and  $b$  are branch vertices of  $K$ . Further, if there exists  $i \in \{1, 2, 3\}$  such that  $x$  is on  $P_i$ , we may assume  $i = 1$ . Let  $p$  be the neighbor of  $x$  on  $P_1$ . Then one of  $u, v$  neighbors at least two of  $q_1, q_2, p$ , say  $u$ . If  $u$  neighbors all three, let  $Q' = aQq_1uq_2Qb, P'_1 = sP_1pu, M' = M \cup \{uv\}$ . Otherwise if  $u$  neighbors  $q_1$  and  $q_2$ , let  $Q' = aQq_1uq_2Qb, P'_1 = sP_1pvwu, M' = M \cup \{uv\}$ . Otherwise, if  $u$  neighbors  $q_1$  and  $p$  or there is no such  $p$ , then let  $Q' = aQq_1uvwq_2Qb, P'_1 = sP_1pu, M' = M$ . Let  $K'$  be found from  $K$  by replacing  $Q$  with  $Q'$ . Then  $(K', M')$  along with  $(s, P'_1, P_2, P_3)$  is a weave and snarl as desired.

So  $x$  is a branch vertex of  $K$ , so has three neighbors in  $K$ ,  $q_1, q_2, q_3$  along segments  $Q_1, Q_2, Q_3$ . Then we may assume  $u$  neighbors  $q_1$  and  $q_2$ . Let  $Q'_i = uq_iQ_i$  for  $i \in \{1, 2\}$ . If  $u$  neighbors all three, let  $Q'_3 = uq_3Q_3$  and  $M' = M \cup \{uv\}$  and  $Q'_3 = uvvq_3Q_3, M' = M$  otherwise. Let  $K'$  be formed from  $K$  by replacing  $Q_i$  with  $Q'_i$  for  $i \in \{1, 2, 3\}$ . If, additionally,  $x$  has a neighbor  $p$  on one of  $P_1, P_2, P_3$ , say  $P_1$ , we define  $P'_1 = sP_1pu$  if  $u$  neighbors  $p$ , otherwise  $P'_1 = sP_1pvwu$  if  $v \in V(M')$  and  $P'_1 = sP_1pv$  if  $v \notin V(M')$ . Otherwise, we let  $P'_1 = P_1$ . Then  $(K', M')$  along with  $(s, P'_1, P_2, P_3)$  gives the desired weave and snarl.

□

If  $G$  is a bipartite graph with bipartition  $(A, B)$ , we often consider subgraphs,  $H$ , of  $G$ , for example that formed from the edges of a skein and snarl. Let  $u$  and  $v$  be vertices of  $G$  with the same parity, say  $A$ , such that there is a segment,  $P$ , containing both  $u$  and  $v$  in  $H$ . Then we say that  $u$  *leans* towards  $v$  (and  $v$  leans towards  $u$ ) and we refer to the process of bicontracting  $B$  vertices of  $P$  until we identify  $u$  and  $v$  as bicontracting  $u$  to  $v$ . By the above argument, if we can find a particular skein in this new bicontracted graph, we can find a similar skein in  $G$  as well. For convenience, if we have a subgraph  $H$  of  $G$  with  $u, x, v \in V(H)$ , then if  $G'$  is the graph formed by bicontracting  $x$  and identifying  $u$  and  $v$  we will also refer to the natural subgraph of  $G'$  formed by this identification as  $H$ .



**Theorem 4.3.2.** *Let  $G$  be a brace with bipartition  $(A, B)$ . Let  $(K, M)$  be a maximal weave of  $G$  with feet  $v_1, v_2, v_3 \in A, v_4, v_5, v_6 \in B$  and segments  $Q_{ij}$  with ends  $v_i$  and  $v_j$ . Then if  $V(K) \cup V(M) \neq V(G)$ ,  $G$  contains a maximal weave  $(K', M')$  and a snarl for  $(K', M')$ ,  $(s, P_1, P_2, P_3)$ , of width 3.*

*Proof.* By Lemma 4.2.4, we may assume  $(K, M)$  contains a snarl  $(s, P_1, P_2, P_3)$  with  $s \in B$  and ends  $x, y, z$ . Let  $P = P_1 \cup P_2 \cup P_3$  and let  $H = K \cup P$ . Further, we may assume that  $y$  and  $z$  lean toward  $v_1$  and  $v_2$  respectively,  $x \in V(Q_{14})$ , and there exists a vertex  $u \in xQ_{14}v_1$  along with an  $M$ -alternating  $K$ -path  $R$  disjoint from  $P$  with ends  $u$  and  $w \notin V(Q_{14} \cup Q_{15} \cup Q_{16})$ . Then  $w$  leans to either  $v_2$  or  $v_3$ .

If  $w$  leans to  $v_2$ , then consider the hex  $K + R - Q_{24}$  (if  $w \notin V(Q_{24})$ ) or  $K + R - v_4Q_{24}w$  (if  $w \in V(Q_{24})$ ). Then  $(s, P_1, P_2, P_3)$  can be converted to a snarl of width three for this hex, and hence the lemma holds. Thus we may assume that  $w$  leans to  $v_3$ .

It follows that  $w \in V(Q_{34} \cup Q_{35} \cup Q_{36})$ . We claim that we may assume that  $w \notin V(Q_{34})$ . Suppose otherwise, so  $w \in V(Q_{34})$ . Consider the graph formed by bicontracting  $y$  to  $v_1$ ,  $z$  to  $v_2$ , and  $w$  to  $v_3$ . Then  $s, v_4$ , and  $u$  are all symmetric, as are  $y, z$ , and  $w$ . In  $G$ , the symmetric statement to  $w$  not on  $Q_{34}$  is that  $y$  is not on  $Q_{14}$  and  $z$  is not on  $Q_{24}$ . So if we had  $y \notin V(Q_{14})$  or  $z \notin V(Q_{24})$ , then by appropriately permuting  $s, v_4, u$  we would find a snarl and weave in which the corresponding  $w$  was not on  $Q_{34}$ . Therefore, we may assume that  $y \in V(Q_{14})$  and  $z \in V(Q_{24})$ . Now choose  $K, M, s, P_1, P_2, P_3, R$  subject to the above so that  $|yQ_{14}v_1| + |zQ_{24}v_2| + |wQ_{34}v_3|$  is as small as possible. Then let  $Y = V(P_1 \cup P_2 \cup P_3 \cup v_4Q_{14}y \cup v_4Q_{24}z \cup v_4Q_{34}w) \cap B$ . We note that the number of neighbors of  $Y$  is  $|Y| + 1$ , so by Lemma 4.2.2, we can find an  $M$ -alternating path with one end in  $Y$  and the other in  $V(G) \setminus (V(M) \cup N(Y))$ . Again by symmetry, we may assume one end of this path lies in  $P$ . If the other end leans to  $v_3$ , we immediately find a width three snarl. If the other end lies on  $Q_{14}$  or  $Q_{24}$ , we may reduce the size of  $|yQ_{14}v_1| + |zQ_{24}v_2| + |wQ_{34}v_3|$ . So the other end lies

on  $Q_{15} \cup Q_{16} \cup Q_{25} \cup Q_{26}$ , which, replacing  $y$  or  $z$  by this new vertex, by the same symmetry argument as before means that there is a choice in which  $w$  does not lie on  $Q_{34}$ . This proves our claim that we may assume that  $w \notin V(Q_{34})$ .

By the symmetry between  $Q_{35}$  and  $Q_{36}$ , we may assume that  $w \in V(Q_{36})$ . We then apply Lemma 4.2.7 with  $v = v_3, P_1 = Q_{36}, P_2 = Q_{35}, P_3 = Q_{34}, q = w$  and  $R$  as  $R$  to find an integer  $i \in \{1, 2\}$  and paths  $P'_1, P'_2, P'_3, R'$  such that one of the three outcomes of that lemma holds. We may assume that  $i = 1, P_j = P'_j$  for  $j \in \{1, 2, 3\}$  and  $R' = R$ .

Suppose we have outcome (1). Then there is a vertex  $p \in V(v_3Q_{36}w)$  along with a path  $T$  and a vertex  $q \in V(K) \cap A$  such that  $T$  has endpoints  $p$  and  $q$ . Then we may assume that  $q$  leans to one of  $v_1, v_2, x$ . Bicontract  $y$  to  $v_1, z$  to  $v_2$ , and  $q$  to the vertex to which it leans. Let  $Z = R \cup T$ . Then, respectively, let  $X$  be defined as  $Q_{24} \cup Q_{16}, vQ_{14}u \cup Q_{23} \cup Q_{34}, Q_{16} \cup Q_{25}$ , in which case the snarls in  $H + Z - X$  with root  $p$  and feet, respectively,  $\{v_1, v_2, x, s, u, v_5\}, \{v_1, v_2, x, s, v_4, v_6\}, \{v_1, v_2, x, s, u, v_4\}$  have width three.

Suppose we have outcome (2). Then there is a vertex  $p \in V(v_3Q_{36}w)$  along with a path  $T$  and a vertex  $q \in V(Q_{35})$  such that  $T$  has endpoints  $p$  and  $q$ . In addition, we have a path  $S$  with endpoints  $r \in V(qQ_{35}v_3)$  and  $t$ . Let  $Z = R \cup T \cup S$ . Then  $t$  leans to one of  $\{v_1, v_2, x\}$ . Let  $X$  be, respectively  $v_1Q_{14}u \cup rQ_{35}q \cup Q_{15}, Q_{16} \cup Q_{24} \cup Q_{25}, v_1Q_{14}u \cup xQ_{14}v_4 \cup Q_{25}$ . Then the snarl in  $H + Z - X$  with roots, respectively  $v_4, u, r$  and feet respectively  $\{v_1, v_2, w, s, p, v_6\}, \{v_2, v_3, q, s, r, p\}, \{v_1, v_2, w, s, p, v_6\}$  have width three.

So we must have outcome (3). Then there is a vertex  $p \in V(v_3Q_{36}w)$  along with a path  $T$  and a vertex  $q \in V(wQ_{36}v_6)$  such that  $T$  has endpoints  $p$  and  $q$ . In addition, we have a path  $S$  with endpoints  $r \in V(wQ_{36}q)$  and  $t$ . Let  $Z = R \cup T \cup S$ . Then  $t$  leans to one of  $\{v_1, v_2, x\}$ . Note that if  $t$  leans to  $v_2$ , replacing  $Q_{36}$  by  $v_3Q_{36}pTqQ_{36}v_6$  and  $R$  by  $uRwQ_{36}rSv_2$ , we have an  $M$ -alternating path between  $u$  and  $v_2$  which is a case

we have already solved. If  $t$  leans to  $v_1$  or  $x$ , let  $X$  be, respectively  $v_1Q_{14}u \cup Q_{16} \cup Q_{25}$ ,  $xQ_{14}u \cup Q_{16} \cup Q_{24}$ . Then the snarl in  $H + Z - X$  with roots respectively  $v_4, v_5$  and feet respectively  $\{v_1, w, q, s, p, r\}, \{x, w, q, s, p, r\}$  have width three.

□

## 4.4 The Main Result

In this section we complete the proof of the main theorem of this chapter:

**Theorem 4.4.1.** *Let  $G$  be a nonplanar brace. Then  $G$  contains one of  $K_{3,3}$ , the Heawood graph, or Rotunda as a matching minor.*

We begin with the following two lemmas that show, in some sense, that  $K_{3,3}$  is the most important of these matching minors. For reference, we include in Figures 4.3 and 4.4 various representations of the Heawood and Rotunda graphs.

**Lemma 4.4.2.** *Let  $G$  be a brace with bipartition  $(A, B)$ . Let  $(K, M)$  be a skein whose pattern is the Heawood graph such that  $V(G) \neq V(K) \cup V(M)$ . Then  $G$  contains a weave  $(K', M')$  of at least the same heft as  $(K, M)$ .*

*Proof.* We view the Heawood graph as an odd hex along with two additional vertices, each of which has three neighbors on segments of the hex. Let the feet of the hex be  $v_1, v_2, v_3 \in A, v_4, v_5, v_6 \in B$  and the segments be  $Q_{ij}$  with ends  $v_i$  and  $v_j$  for  $i \in \{1, 2, 3\}$  and  $j \in \{4, 5, 6\}$ . Let the two vertices not in the hex be  $s_1 \in A, s_2 \in B$ . The neighbors of  $s_1$  are  $x_1, x_2, x_3$ , the neighbors of  $s_2$  are  $y_1, y_2, y_3$  and path  $P_i$  has ends  $s_1$  and  $x_i$  and  $R_i$  has ends  $s_2$  and  $y_i$ . Finally,  $x_1 \in V(Q_{14}), x_2 \in V(Q_{25}), x_3 \in V(Q_{36})$ . We note also that the Heawood graph is vertex-transitive.

If  $V(G) \neq V(K) \cup V(M)$ , we can find a vertex  $t \in V(G) \setminus (V(K) \cup V(M))$  along with an  $M$ -alternating paths  $S$  with ends  $t$  and  $z \in V(K)$ . By definition,  $z$  leans towards one of the vertices of  $V(K)$ , so, since they are all the symmetric, we may assume it is  $s_1$  and, by Lemma 4.3.1, we may assume  $z = s_1$ . But then we consider

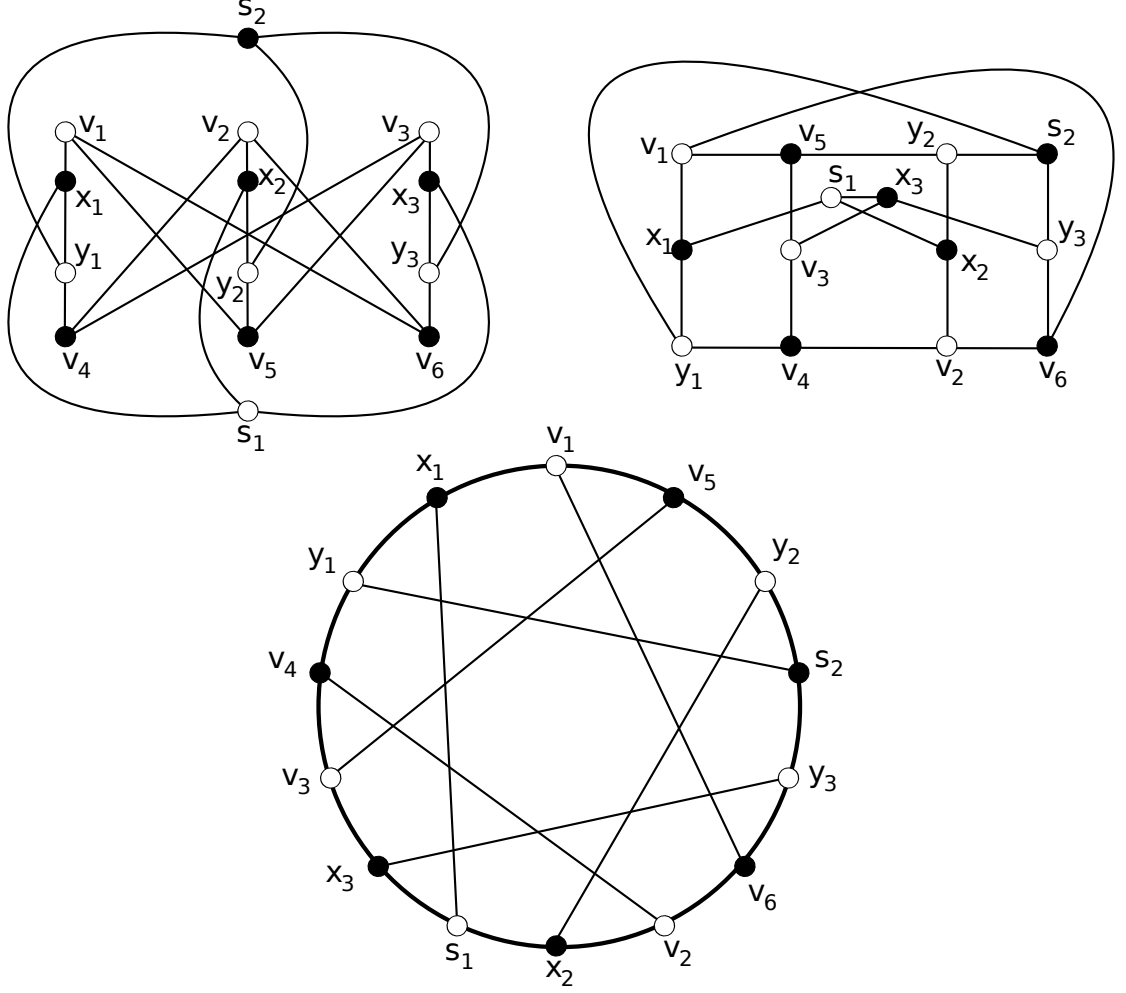


Figure 4.3: Three drawings of the Heawood graph

the hex defined above with feet  $v_i, 1 \leq i \leq 6$  with  $M$  grown maximally to find a weave of the same heft as  $(K, M)$ .  $\square$

**Lemma 4.4.3.** *Let  $G$  be a brace with bipartition  $(A, B)$ . Let  $(K, M)$  be a skein whose pattern is Rotunda. Then if  $V(K) \cup V(M) \neq V(G)$ ,  $G$  contains a weave  $(K', M')$  of at least the same heft as  $(K, M)$ .*

*Proof.* We view Rotunda as a four-separation along with three vertex disjoint 4-cycles. We will refer to the 4-cycles as *cores* and to the vertices of the four-separation as *pivots*. We label the branch vertices corresponding to the vertices of the 4-cycles

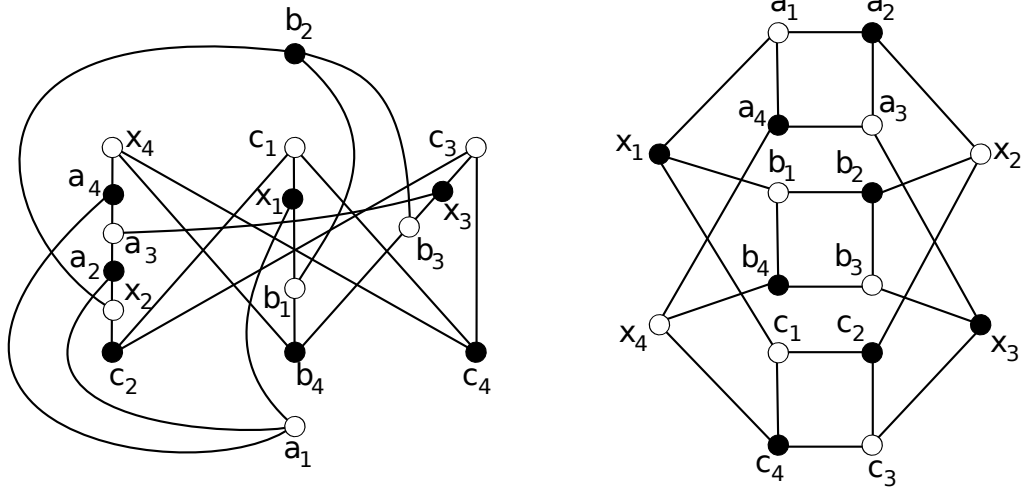


Figure 4.4: Two drawings of Rotunda

in order as  $a_i, b_i$ , or  $c_i$ ,  $1 \leq i \leq 4$  with  $a_1$  and  $a_3 \in A$  and  $a_2$  and  $a_4 \in B$  and similarly for the others. The pivots are  $x_1, x_2, x_3, x_4$  where there is a segment between  $x_i$  and  $a_i, b_i$ , and  $c_i$  for  $1 \leq i \leq 4$ . The segment from  $x_i$  to  $v_i$ ,  $1 \leq i \leq 4, v \in a, b, c$  is  $Q_{vi}$  and the segment between  $v_i$  and  $v_{i+1}$  is  $P_{vi}$  where the addition is taken modulo 4.

Since  $V(K) \cup V(M) \neq V(G)$ , we can find a vertex  $t \in V(G) \setminus (V(K) \cup V(M))$  along with 2  $M$ -alternating paths,  $T_1, T_2$  with ends  $t$  and respectively  $z_1, z_2 \in V(K)$ . Note that by Theorem 4.2.5, we can choose  $t$  and the paths so that  $z_1$  and  $z_2$  do not both lean to the same vertex. We may assume that  $t \in A$ . Note that if either  $z_1$  or  $z_2$  leans towards  $b_2$ , we may assume  $z_1 = b_2$  by Lemma 4.3.1. Then let  $X = P_{a1} \cup P_{a4} \cup Q_{a1} \cup Q_{a3} \cup Q_{b2} \cup P_{b1} \cup P_{b2}$  in which case  $K - X$  with feet  $x_4, c_1, c_3, c_2, b_4, c_4$  gives a weave with the same heft as  $(K, M)$ .

By symmetry, then, we must have that  $z_1$  and  $z_2$  lean towards, respectively,  $x_1$  and  $x_3$ , so, by Lemma 4.3.1, we may assume that  $z_1 = x_1$  and  $z_2 = x_3$ . Let  $T = T_1 \cup T_2$ . Let  $X$  be the union of the segments of  $K$  incident with  $\{a_1, a_2, a_3, a_4, c_3\}$ . Then  $K + T - X$  is a hex with feet  $s_2, x_3, v_5, y_2, y_3, v_2$  and the corresponding weave has the same heft as  $(K, M)$ .

□

We are now prepared to prove the main result.

**Theorem 4.4.4.** *Let  $G$  be a brace. Suppose  $G$  contains an odd subdivision of  $K_{3,3}$ . Then  $G$  contains a subgraph  $K$  isomorphic to an odd subdivision of either  $K_{3,3}$ , Rotunda, or the Heawood graph with a perfect matching in the complement.*

*Proof.* Choose  $K, M$  so that  $K$  is an odd subdivision of  $K_{3,3}$ , rotunda, or the Heawood graph, and  $M$  is a maximal matching in its complement. Let the bipartition of  $G$  be  $(A, B)$ . Select  $K, M$  so that  $X = V(G) \setminus V(K \cup M)$  is as small as possible and, subject to that,  $K$  is a hex if possible. If  $|X| = 0$ , we are done. Note that  $|X|$  is even and that there are the same number of  $A$  and  $B$  vertices in  $X$ .

Note that by our choice of  $K$  and Lemmas 4.4.2 and 4.4.3, we may assume that  $K$  is a hex. Let the feet of  $K$  be  $v_1, v_2, v_3 \in A, v_4, v_5, v_6 \in B$  and the segments be  $Q_{ij}$  with ends  $v_i$  and  $v_j$ . Then by Theorem 4.3.2,  $G$  contains a snarl  $(s, P_1, P_2, P_3)$  with  $s \in B$  and ends  $x, y, z$  with  $x \in V(Q_{14} \cup Q_{15} \cup Q_{16}), y \in V(Q_{24} \cup Q_{25} \cup Q_{26})$  and  $z \in V(Q_{34} \cup Q_{35} \cup Q_{36})$ .

We view the structure formed by  $K$  and this snarl as a skein (with the matching changed in the natural way),  $(K', M')$ . Then by Lemma 4.2.1 and Lemma 4.2.3, there exists a snarl with respect to this skein, not all of whose ends are on a path in  $K'$  with all interior vertices of degree 2 in  $K'$ . Let this second snarl be  $(t, R_1, R_2, R_3)$ ,  $t \in A$  with ends  $a, b, c$ . Note that the ends of this snarl are not in  $P_1 \cup P_2 \cup P_3$  since otherwise we would have an  $M$ -alternating path between  $s$  and  $t$  which contradicts the maximality of  $(K, M)$ . Let  $H = K \cup P_1 \cup P_2 \cup P_3 \cup R_1 \cup R_2 \cup R_3$ .

Suppose there exists  $i \in \{1, 2, 3\}, j \in \{4, 5, 6\}, p \in \{a, b, c\}$  so that  $v_j Q_{ij} p \cap \{x, y, z\}$  is empty. Then we may assume  $i = 1, j = 4, p = a$ . Then by Lemma 4.3.1 we may assume  $x = v_1, y = v_2, z = v_3$ , and  $a = v_4$ . Let  $Y = Q_{14} \cup Q_{24} \cup Q_{34} \cup R_1 \cup R_2 \cup R_3$ . Then  $H - Y$  is a hex with feet  $v_1, v_2, v_3, s, v_5, v_6$  and the resulting weave has greater heft than  $(K, M)$ .

So we may assume that  $a, b, c$  fall on a subset of the segments on which  $x, y, z$  fall.

Suppose there exists  $i_1, i_2 \in \{1, 2, 3\}, j \in \{4, 5, 6\}$ ,  $i_1 \neq i_2$  such that one of  $a, b, c$  lies on  $Q_{i_1 j}$  and another on  $Q_{i_2 j}$ . We may assume  $i_1 = 1, i_2 = 2, j = 4$  and  $a$  lies on  $Q_{14}$ ,  $b \in V(Q_{24})$ . Then we may assume  $x \in V(v_4 Q_{14} a)$ ,  $y \in V(v_4 Q_{24} b)$ . By Lemma 4.3.1, we may assume  $z = v_3$ . Let  $Y$  be  $R_3 \cup Q_{25} \cup Q_{26} \cup v_2 Q_{24} b \cup Q_{15}$ . Then  $H - Y$  is a hex with feet  $s, a, v_4, x, y, v_3$  and the resulting weave has greater heft than  $(K, M)$ .

There are then (up to symmetries) only two options for which of the  $Q_{ij}$   $a, b, c$  are on. Either they land on  $Q_{14}, Q_{25}, Q_{36}$  up to symmetry, or  $a$  and  $b$  are on  $Q_{14}$  and  $c$  is on  $Q_{25}$ . The first case, along with  $P_1, P_2, P_3$  is exactly a subdivision of the Heawood graph which gives a skein of larger heft. So we may assume the second case.

So we have  $Q_{14} = v_1 \dots a \dots b \dots x \dots v_4$  and  $Q_{25} = v_2 \dots c \dots y \dots v_5$  and  $z$  is on  $Q_{3j}$  for some  $j \in 4, 5, 6$ . Let  $T$  be the set of vertices in  $A$  reachable from  $t$  by  $M$ -alternating paths. Note that by the preceding arguments, all of the neighbors of  $T$  lie on  $xQ_{14}v_1 \cup yQ_{25}v_5 \cup zQ_{3j}v_3$ . For  $i \in \{1, 2, 3\}, j \in \{4, 5, 6\}$ , let  $a_{ij}$  be the neighbor of  $T$  closest to  $v_i$  on  $Q_{ij}$  and  $b_{ij}$  the neighbor of  $T$  closest to  $v_j$  on  $Q_{ij}$ . Let  $\phi_{ij} = |v_i Q_{ij} a_{ij}| + |v_j Q_{ij} b_{ij}|$  if  $a_{ij}$  and  $b_{ij}$  exist and are distinct and  $|Q_{ij}|$  otherwise. Similarly, let  $W_{ij} = a_{ij} Q_{ij} b_{ij}$  if  $a_{ij}$  and  $b_{ij}$  exist and are distinct and  $\emptyset$  otherwise. Let  $\phi = \sum_{i \in \{1, 2, 3\}, j \in \{4, 5, 6\}} \phi_{ij}$  and  $W = \bigcup W_{ij}$ . Choose  $K, s, t$  subject to the above so that  $\phi$  is minimum.

Let  $W_A = W \cap A$  and let  $T^* = T \cup W_A$ . Then the number of neighbors of  $T^*$  is at most  $|T| + 1$  since at most two of the  $W_{ij}$  are non-empty. So we apply Lemma 4.2.2 to find a vertex  $u \in T^*$  and an  $M$ -alternating path  $S$  with ends  $u$  and  $w \in V(G) \setminus (N(T^*) \cup V(M))$ . Note that  $u$  is not in  $T$  by our choice of  $T$  (since otherwise we can either find an  $M$ -alternating path between two vertices unmatched under  $M$  or decrease  $\phi$ ). Then without loss of generality, we may assume that  $u \in V(Q_{14})$ ,  $w \notin V(Q_{14})$  (since otherwise we replace  $Q_{14}$  by  $v_4 Q_{14} u S w Q_{14}$  and reduce  $\phi$ ) and that we can find  $t, R_1, R_2, R_3$  with ends  $a, b, c$  as described above and disjoint from  $S$ .

Note that if we ignore  $R_3$  and  $S$ ,  $u$  and  $t$  are symmetric. Since the end of  $R_3$  must lie on either  $yQ_{25}v_2$  or on the segment containing  $z$  (so long as it is not  $Q_{34}$ ) as argued

above, the same must be the case for the end of  $S$ . Note that  $w \in zQ_{35}v_5$  is exactly a subdivision of Rotunda which gives a skein of larger heft. If  $w \in yQ_{25}v_2$ , by Lemma 4.3.1, we may assume  $z = v_3$  and  $w = c$ . The two cases are  $w \in yQ_{25}v_2, w \in zQ_{36}v_6$ . Let  $Z$  be, respectively,  $v_4Q_{14}x \cup Q_{24} \cup Q_{34} \cup P_3 \cup Q_{34} \cup Q_{35} \cup Q_{36} \cup Q_{16} \cup Q_{26} \cup v_2Q_{25}c$  or  $R_1 \cup v_3Q_{36}w \cup Q_{16} \cup Q_{24} \cup bQ_{14}u$ . Then the hex  $H + S - Z$  with feet  $\{u, y, t, a, b, c\}, \{s, c, v_5, x, y, z\}$  respectively gives rise to a weave of larger heft which is a contradiction.

□

## 4.5 Rotunda and Heawood

One of the outcomes of Theorem 4.1.2 is a perfect Rotunda or Heawood-skein. We would like to better understand these outcomes; specifically, we would like to understand when they contain perfect weaves. We note that these results are also contained in [42] in which they are used to develop the algorithm to find a Pfaffian orientation for a bipartite graph, though, particularly in the case of Rotunda, the proofs contained there are significantly more difficult.

**Definition.** Let  $(K, M)$  be an  $H$ -skein and let  $R$  be an  $M$ -alternating path with ends  $u, v \in V(K)$ , otherwise vertex disjoint from  $K$ . Let  $x$  and  $y$  be the vertices of degree at least 3 in  $K$  such that  $u$  leans to  $x$  and  $v$  to  $y$ . Then we say that  $R$  is a *skip* if there is no path between  $x$  and  $y$  in  $K$  with all the interior vertices of degree exactly 2. The *ends* of the skip are  $x$  and  $y$ .

We will be discussing the Heawood graph in several of the following lemmas. In the interest of conserving space, we use the following canonical notation unless we specify otherwise. We view the Heawood graph as an odd hex along with two additional vertices, each of which has three neighbors on segments of the hex. Let the feet of the hex be  $v_1, v_2, v_3 \in A, v_4, v_5, v_6 \in B$  and the segments be  $Q_{ij}$  with ends  $v_i$  and  $v_j$  for  $i \in \{1, 2, 3\}$  and  $j \in \{4, 5, 6\}$ . Let the two vertices not in the hex be  $s_1 \in A, s_2 \in B$ .



The neighbors of  $s_1$  are  $x_1, x_2, x_3$ , the neighbors of  $s_2$  are  $y_1, y_2, y_3$  and path  $P_i$  has ends  $s_1$  and  $x_i$  and  $R_i$  has ends  $s_2$  and  $y_i$ . Finally,  $x_1 \in V(Q_{14}), x_2 \in V(Q_{25}), x_3 \in V(Q_{36})$ .

Further, note that in the Heawood graph every vertex of the same parity is symmetric, as is every pair of non-adjacent vertices of opposite parity. This leads to the following lemma for the Heawood graph:

**Lemma 4.5.1.** *Let  $G$  be a brace with bipartition  $(A, B)$  and  $(K, M)$  a perfect Heawood-skein in  $G$  with a skip  $R$ . Then  $G$  contains a perfect weave.*

*Proof.* By the symmetries mentioned above, we may assume that the skip is between  $s_1$  and  $s_2$ . But then the hex with feet  $v_i$  and paths  $Q_{ij}$  and the corresponding weave completes the proof.  $\square$

**Lemma 4.5.2.** *Let  $G$  be a brace with bipartition  $(A, B)$  and  $(K, M)$  a perfect Heawood-skein in which  $K$  is not the Heawood graph. Then  $G$  contains a perfect weave.*

*Proof.* By symmetry and Lemma 4.2.6, we find a vertex  $u \in A$  on  $P_1$  along with an  $M$ -alternating path  $R$  to a vertex leaning to one of the  $B$  vertices of  $K$  other than  $x_1$ . If  $R$  is a skip, we are done by Lemma 4.5.1 so we may assume the ends of  $R$  are  $u$  and  $w$  which leans to  $x_2$ . We now apply Lemma 4.2.7 with  $v = s_1, a = x_1, b = x_3, c = x_2, p = x_2, q = w$  and  $R$  serving as  $R$  to find one of the outcomes of that lemma.

Suppose we have outcome (i). Then there is a vertex  $p \in V(s_1 P_1 u)$ , a path  $T$  with ends  $p$  and  $q \in V(K)$ . If  $q$  leans to a vertex other than  $y_1$  or  $v_1$ ,  $R$  is a skip, so we are done. So we may assume  $q$  leans to  $y_1$  or  $v_1$ ; again by symmetry we may assume  $y_1$ . By Lemma 4.3.1, we may assume  $q = y_1$  and  $w = x_2$ . Let  $X = T \cup R$  and  $Y = P_3 \cup v_3 Q_{36} x_3 \cup Q_{24} \cup v_2 Q_{25} x_2 \cup v_5 Q_{25} y_2 \cup v_4 Q_{14} y_1 \cup Q_{34} \cup Q_{35} \cup Q_{26} \cup Q_{15}$ . Then  $K + X - Y$  is a hex with feet  $x_1, p, c, y_1, u, s_1$  and the corresponding weave is perfect.

Suppose we have outcome (ii). Then there is a vertex  $p \in V(s_1 P_1 u)$  and a path  $T$  with ends  $p$  and  $q \in uV(P_3)$ , along with  $r \in V(q P_3 s_1)$  and a path  $S$  to  $t \in V(K)$ .

Then  $t$  leans to  $y_3$  or  $v_3$  since otherwise  $S$  is a skip. Then we replace  $s_1$  by  $u$ , taking  $P'_1 = uP_1x, P'_2 = R, P'_3 = uP_1pTqP_3x_3$  and take  $R' = x_2P_2s_1P_3rSt$  in which case  $R'$  is a skip.

Suppose we have outcome (iii). Then there is a vertex  $p \in V(s_1P_1u)$  and a path  $T$  with ends  $p$  and  $q \in uP_1x_1$ , along with  $r \in V(uP_1p)$  with a path  $S$  to  $t \in V(K)$ . Then either  $S$  is a skip or we reroute  $P_1$  to be  $s_1P_1pTqP_1x_1$  and take  $R' = x_2RuP_1rSt$  in which case  $R'$  is a skip.  $\square$

This allows us to prove the main result for the Heawood graph:

**Theorem 4.5.3.** *Let  $G$  be a brace with bipartition  $(A, B)$  not isomorphic to the Heawood graph and  $(K, M)$  a perfect Heawood-skein. Then  $G$  contains a perfect weave.*

*Proof.* By Lemma 4.5.2, we may assume  $K$  is isomorphic to the Heawood graph. Let  $a, b$  be two vertices matched under  $M$  with  $a \in A, b \in B$ . Then it is easy to see using the brace property as in the proof of Lemma 4.2.1 that there are at least four distinct vertices of  $K$ , two in  $A$  and two in  $B$ , reachable by  $M$ -alternating paths from either  $a$  or  $b$ . Since the Heawood graph has girth 6, there must be two of these that are non-adjacent in  $K$ , say  $u \in A$  and  $w \in B$ . Let  $S_1$  be the  $M$ -alternating path between  $a$  and  $w$  and  $S_2$  the  $M$ -alternating path between  $b$  and  $u$ . Let  $a'$  be the vertex of  $S_2$  closest to  $w$  along  $S_1$  or  $a$  if no such vertex exists. Let  $b'$  be the vertex matched to  $a'$  and  $e$  the edge between  $a'$  and  $b'$ . Note that  $b' \in V(S_2)$ . Then  $wS_1a'eb'S_2u$  is a skip, so  $G$  contains a perfect weave.  $\square$

We now move onto Rotunda. As in the case of the Heawood graph, it will be convenient to have a canonical labeling for a Rotunda-skein. We view Rotunda as a four-separation along with three vertex disjoint 4-cycles. We will refer to the 4-cycles as *cores* and to the vertices of the four-separation as *pivots*. We label the branch vertices corresponding to the vertices of the 4-cycles in order as  $a_i, b_i$ , or  $c_i$ ,  $1 \leq i \leq 4$  with  $a_1$  and  $a_3 \in A$  and  $a_2$  and  $a_4 \in B$  and similarly for the others. The pivots are

$x_1, x_2, x_3, x_4$  where there is a segment between  $x_i$  and  $a_i, b_i$ , and  $c_i$  for  $1 \leq i \leq 4$ . The segment from  $x_i$  to  $v_i$ ,  $1 \leq i \leq 4$ ,  $v \in a, b, c$  is  $Q_{vi}$  and the segment between  $v_i$  and  $v_{i+1}$  is  $P_{vi}$  where the addition is taken modulo 4.

**Lemma 4.5.4.** *Let  $G$  be a brace with bipartition  $(A, B)$  and  $(K, M)$  a perfect Rotunda-skein in  $G$  with a skip  $R$  with at least one end not a pivot. Then  $G$  contains a perfect weave.*

*Proof.* By symmetry, we may assume one end of  $R$  is  $a_1$  and the other is  $b_2$  or  $x_3$ . Then let  $X$  be, respectively,  $Q_{a1} \cup Q_{a2} \cup Q_{a3} \cup P_{a1} \cup P_{a2} \cup P_{a3} \cup P_{b1} \cup P_{b2}$ ,  $Q_{a1} \cup Q_{b1} \cup Q_{c1} \cup Q_{b3} \cup Q_{c4} \cup P_{b1} \cup P_{b4} \cup P_{c2}$  with feet  $\{x_4, c_1, c_3, c_2, b_4, c_4\}, \{a_1, a_3, x_2, a_2, a_4, x_3\}$  respectively. Then  $K + R - X$  is a hex which gives rise to a perfect weave.

□

To continue, we will need the following extension of Lemma 4.2.7.

**Lemma 4.5.5.** *Let  $G$  be a brace with bipartition  $(A, B)$ , let  $H$  be a graph with no vertices of degree two, let  $(K, M)$  be a maximal  $H$ -skein in  $G$ , let  $v \in A$  be a branch-vertex of  $K$  of degree three, let  $P_1, P_2, P_3$  be the three segments of  $K$  incident with  $v$ , and let  $v_1, v_2, v_3$ , respectively, be their other ends. Let  $Q_1, Q_2$  be distinct members of  $P_1, P_2, P_3$  and let  $q \in A \cap V(Q_1)$ , let  $p \in B \cap V(Q_2)$ , and let  $R$  be an  $M$ -alternating  $K$ -path in  $G$  with ends  $p$  and  $q$ . Then there exists a maximal  $H$ -skein  $(K', M')$ , where  $K'$  is obtained from  $K$  by replacing the paths  $P_1, P_2, P_3$  by paths  $P'_1, P'_2, P'_3$ , respectively, and  $P'_i$  has ends  $v'$  and  $v_i$ . Furthermore, there exist distinct integers  $i, j$  in  $\{1, 2, 3\}$ , vertices  $q' \in V(P_i) \cap A$  and  $p' \in V(P'_j) \cap B$ , an  $M$ -alternating  $K'$ -path  $R'$  with ends  $p'$  and  $q'$  and vertices  $u_1 \in vP'_i q' \cap B$ ,  $u_2 \in V(K') \cap A - V(P'_1 \cup P'_2 \cup P'_3)$  and an  $M'$ -alternating  $(K' \cup R')$ -path with ends  $u_1$  and  $u_2$ .*

*Proof.* Let  $X = V(K) \cap B - V(P_1 \cup P_2 \cup P_3)$ . Let  $P$  be the element of  $\{P_1, P_2, P_3\}$  containing  $q$ . By Lemma 4.2.6, there exists an augmenting sequence  $R_1, \dots, R_n$

subject to  $P$  and  $M$  such that the ends of  $R_i$  are  $u_{2i-1}$  and  $u_{2i}$  and  $u_{2n} \in V(K) \cap B - V(P_1 \cup P_2 \cup P_3)$ . Choose  $v, P_1, P_2, P_3, M$  and such an augmenting sequence so that the augmenting sequence has as few augmentations as possible and, subject to that, such that as few of the ends of the augmentations lie on  $P_3$  as possible. Note that only  $u_1 \in V(vPq)$  since otherwise we immediately have a shorter augmenting sequence. Therefore, if the sequence has length 1, we are done, so we may assume it has length at least 2.

We may assume without loss of generality that  $Q_1 = P_1, Q_2 = P_2$ .

Suppose first that  $u_2 \in V(R)$ . Then take  $v' = q, P'_1 = qP_1v_1, P'_2 = qRpP_2v_2, P'_3 = qP_1vP_3v_3, p' = u_1, q' = u_2, u'_1 = u_3, R' = R_1$  to find a shorter augmenting sequence under a permutation of  $\{v_1, v_2, v_3\}$ . Suppose  $u_2 \in V(P_2 \cup P_3)$ . Then take  $p' = u_1, q' = u_2, R' = R_1$  to find a shorter sequence. Finally, take  $u_2 \in V(qP_1v_1)$ . Then take  $v' = p, P'_1 = pP_1v_1, P'_2 = pRqP_2v_2, P'_3 = pP_1vP_3v_3, p' = u_1, q' = u_2, R' = R_1, u'_1 = u_3$  to find a shorter augmenting sequence as desired.  $\square$

**Definition.** Let  $(K, M)$  be a Rotunda-skein and let  $P$  be a path between two components of  $K \setminus \{x_1, x_2, x_3, x_4\}$  vertex disjoint from  $K$  except at its ends. Then we refer to  $P$  as a *link*.

We now show that perfect rotunda-skeins with  $M$ -augmenting links contain perfect weaves.

**Lemma 4.5.6.** *Let  $G$  be a brace with bipartition  $(A, B)$  and  $(K, M)$  a perfect Rotunda-skein in  $G$  with an  $M$ -alternating link  $R$ . Then  $G$  contains a perfect weave.*

*Proof.* We may assume that  $R$  is either not a skip or the ends of  $R$  lean to two pivots since otherwise we are done by Lemma 4.5.4. If both ends of the link lean to core vertices, we immediately find a skip. So we may assume at least one of the ends of the link,  $u$  on  $Q_{a1}$  leans to a pivot. Then either the other end,  $w$ , leans to a core vertex or to a pivot. We first consider the case in which  $u$  leans to  $x_1$  and  $w$  to a core

vertex, either  $b_1$  or  $c_1$ , say  $b_1$ . We may then apply either Lemma 4.2.7 or Lemma 4.5.5 depending on whether  $w$  is on a core path or on  $P_{b_1}$ . We note that if we apply Lemma 4.3.1 after applying one of these Lemmas and assume  $w = b_1$ , then the outcome of Lemma 4.5.5 is the same as outcome (1) of Lemma 4.2.7 which we will show contains a perfect weave. So we may apply Lemma 4.2.7 with  $v = x_1, q = u, p = w, a = a_1, b = c_1, c = b_1, P_1 = Q_{a_1}, P_2 = Q_{c_1}, P_3 = Q_{b_1}$  and  $R = R$  to find one of the outcomes of that lemma. In any of the outcomes, we find a vertex  $v_1 \in V(x_1 Q_{a_1} u \cap A)$  and an  $M$ -augmenting path  $R_1$  with ends  $v_1$  and  $v_2$ , where  $v_2$  is defined by the outcome.

Suppose we have outcome (1). Then  $v_2$  leans to either  $a_2$  or  $a_4$  since otherwise we immediately have an appropriate skip. These two cases are symmetric, so we assume  $v_2$  leans to  $a_2$ . By Lemma 4.3.1 we may assume  $v_2 = a_2$  and  $w = b_1$ . Then let  $S = R_1 \cup R$  and let  $X$  be  $Q_{b_4} \cup Q_{a_2} \cup Q_{b_2} \cup Q_{c_2} \cup Q_{c_3} \cup P_{a_3} \cup P_{b_1} \cup P_{b_2} \cup P_{c_4}$  in which case  $K + S - X$  is a hex with feet  $u, a_2, x_1, v_1, a_1, b_1$  which gives rise to a perfect weave.

Suppose we have outcome (2). Then  $v_2$  lies on  $u Q_{a_1} a_1$ , and we have  $v_3 \in V(u Q_{a_1} v_3)$  and a path  $R_2$  with ends  $v_3$  and  $v_4$ . Since  $R_2$  is not a skip, we may assume that  $v_4$  leans to  $a_2$  or  $a_4$ . But then rerouting  $Q_{a_1}$  as  $x_1 Q_{a_1} v_1 R_1 v_2 Q_{a_1}$  gives a skip  $b_1 R u Q_{a_1} v_3 R_2$ .

Suppose we have outcome (3). Then we may assume  $v_2$  lies on  $Q_{c_1}$ . Then  $v_3 \in V(x_1 Q_{c_1} v_2)$  and, by symmetry,  $v_4$  leans to  $c_4$ . But then this is exactly the case of the length one augmenting path by viewing  $v_2$  as  $u$ ,  $v_1$  as  $w$ ,  $v_3$  as  $v_1$  and  $v_4$  as  $v_2$ .

So we may assume that  $R$  is a skip between two pivots. Then we may assume that  $w$  is on  $Q_{b_4}$ . We then apply Lemma 4.2.7 with  $v = x_1, p = w, q = u, a = a_1, b = c_1, c = b_1, P_1 = Q_{a_1}, P_2 = Q_{c_1}, P_3 = Q_{b_1}$  and  $R = R$  to find one of the outcomes of that lemma. In any of the outcomes, we find a vertex  $v_1 \in V(x_1 Q_{a_1} u \cap A)$  and an  $M$ -augmenting path  $R_1$  with ends  $v_1$  and  $v_2$ , where  $v_2$  is defined by the outcome.

Suppose we have outcome (2). Then  $v_2 \in V(u Q_{a_1} a_1)$ . Then  $v_3 \in V(u Q_{a_1} v_2)$  and  $v_4$  leans to  $a_2$  or  $a_4$ . Suppose first that  $v_2 \in V(u Q_{a_1} a_1)$ . Then  $v_3 \in V(u Q_{a_1} v_2)$  and

$v_4$  leans to  $a_2$  or  $a_4$ . If it leans to  $a_2$ , then rerouting  $Q_{a_1}$  as  $a_1Q_{a_1}v_2R_1v_1Q_{a_1}x_1$  gives a skip  $v_4R_2v_3Q_{a_1}uRx_4$ . So we may assume  $v_4$  leans to  $a_4$ . But then rerouting  $Q_{a_1}$  as  $a_1Q_{a_1}v_2R_1v_1Q_{a_1}x_1$  gives an  $M$ -alternating link  $v_4R_2v_3Q_{a_1}uRx_4$  not between two pivots which we showed above gives the desired result.

Suppose we have outcome (3). Then we may assume  $v_2 \in V(Q_{b_1} \cap B)$  in which case  $R_1$  is a skip not between two pivots.

Suppose we have outcome (1). Then  $v_2$  must lean to  $a_4$  or  $a_2$  or else  $R_1$  is a skip not between two pivots. Suppose  $v_2$  leans to  $a_4$ . Then by Lemma 4.3.1 we may assume  $v_2 = a_4$  and  $w = x_4$ . Let  $S = R \cup R_1$  and let  $X$  be  $Q_{b_4} \cup Q_{c_2} \cup Q_{a_3} \cup Q_{b_3} \cup Q_{c_3} \cup P_{a_2} \cup P_{a_3} \cup P_{b_1} \cup P_{c_4}$  in which case  $K + S - X$  is a hex with feet  $u, a_4, x_1, v_1, a_1, x_4$  and the corresponding weave is perfect.

So we have that  $v_2$  leans to  $a_2$ . Note that by applying the same argument to the  $wQ_{b_4}x_4$  piece, we can find a path  $R_2$  with ends  $r$  and  $s$  with  $r \in wQ_{b_4}x_4$ ,  $s$  leaning to  $b_3$ . Then by Lemma 4.3.1 we may assume  $s = b_3$  and  $v_2 = a_2$  and let  $X$  be  $Q_{b_1} \cup wQ_{b_4}r \cup Q_{c_4} \cup Q_{a_2} \cup Q_{b_2} \cup Q_{c_2} \cup Q_{b_3} \cup P_{b_4} \cup P_{c_1} \cup P_{c_2}$ . Let  $S = R \cup R_1 \cup R_2$ . Then  $K + S - X$  is a hex with feet  $a_1, a_3, v_1, a_2, a_4, u$  and the corresponding weave is perfect.

□

We are now prepared to answer the question of when graphs with perfect Rotunda-skeins have perfect weaves.

**Theorem 4.5.7.** *Let  $G$  be a brace with bipartition  $(A, B)$  and  $(K, M)$  a perfect Rotunda-skein in  $G$  with a link  $R$ . Then  $G$  contains a perfect weave.*

*Proof.* If  $R$  is  $M$ -alternating, then we are done. We refer to a vertex of  $R$  as matched if its neighbor under  $M$  is a neighbor in  $R$  and unmatched otherwise. Choose  $R$  to have as few unmatched vertices as possible and, subject to that, has as many ends that lean to core vertices as possible.

Then we may assume one end of  $R$  is  $u \in V(x_1Q_{a_1}a_1)$  by symmetry and that it leans to a core vertex if either end of  $R$  leans to a core vertex. Let the other end of  $R$  be  $r$ . Let  $p$  be the closest unmatched vertex in  $R$  to  $u$  and  $q$  its neighbor under  $M$ . By the brace property, we can find two  $M$ -alternating paths,  $S_1, S_2$  from  $q$  to  $K \cup R$  avoiding  $p$ , not necessarily vertex-disjoint, with ends  $s_1, s_2$  respectively. If either lies in  $R$ , say  $S_1$  we can replace the subpath of  $r$  between  $p$  and  $s_1$  by  $pqS_1$  to find a link with fewer unmatched vertices. If either lies on a path not incident with the core containing  $a_1$ , say  $S_1$ , then  $uRpqS_1$  is an  $M$ -alternating link. If either  $s_1$  or  $s_2$  leans to a core vertex, say  $s_1$ , then if  $u \in A$   $uRpqS_1$  is a skip with an end that is not a pivot and otherwise  $s_1S_1qpRr$  is a link with the same number of unmatched vertices, but with an end that leans to a core vertex. So we may assume  $s_1$  and  $s_2$  both lean to pivots. If  $s_1$  or  $s_2$  is not a pivot, say  $s_1$ , then either  $uRpqS_1$  is an  $M$ -alternating link or  $rRpqS_1$  is a link with fewer unmatched vertices. So  $s_1$  and  $s_2$  are the two pivots of opposite parity from  $u$ . Then if  $u \in A$ ,  $uRpqS_1$  is a skip with an end that is not a pivot, so we may assume  $u \in B$ . So we have  $s_1 = x_4$  and  $s_2 = x_2$ .

We then apply Lemma 4.2.7 with  $v = x_1, a = a_1, b = b_1, c = c_1, p = p, q = u, P_1 = Q_{a_1}, P_2 = Q_{a_2}, P_3 = Q_{a_3}$  and  $uRp$  as  $R$ . We find one of the three outcomes of that lemma. Then we have a vertex  $v_1 \in V(x_1Q_{a_1}u) \cap A$ , a vertex  $v_2$  determined by the outcome in question, and an  $M$ -alternating path  $R_1$  between them. Note that the symmetry between  $Q_{a_1}$  and  $Q_{a_2}$  allows us to choose  $v_1 \in V(Q_{a_1})$ .

Suppose we have outcome (1). Then  $v_2 \in V((S_1 \cup S_2 \cup R \cup K) \setminus (Q_{a_1} \cup Q_{a_2} \cup Q_{a_3} \cup uRp)) \cap B$ . If  $v_2$  lies on one of  $R, S_1, S_2$ , we have a link with the same or fewer unmatched vertices with one end leaning to a core vertex, respectively  $v_1R_1v_2Rr, v_1R_1v_2S_1s_1, v_1R_1v_2S_2s_2$ . So  $v_2$  leans to  $a_2$  or  $a_4$  and, by symmetry, we may assume it leans to  $a_4$ . Then by Lemma 4.3.1 we may assume  $v_2 = a_4$  and let  $X$  be  $Q_{a_2} \cup Q_{b_2} \cup Q_{c_2} \cup Q_{c_3} \cup P_{a_3} \cup P_{b_1} \cup P_{b_2} \cup P_{c_4}$ . Then  $K + S_1 - X$  is a hex with feet  $u, a_4, x_1, v_1, a_1, x_4$  and gives rise to a perfect weave.

Suppose we have outcome (2). Then  $R_1$  is an  $M$ -alternating link.

Suppose we have outcome (3). Then  $v_2 \in V(uQ_{a_1}a_1)$ . Further, there exists  $v_3 \in V(uQ_{a_1}v_2)$  and  $v_4 \in V((S_1 \cup S_2 \cup R \cup K) \setminus (Q_{a_1} \cup Q_{a_2} \cup Q_{a_3} \cup uRp) \cap B$  with a path  $R_2$  between them. Again, if  $v_4$  lies on one of  $R, S_1, S_2$ , then we have a link with the same or fewer unmatched vertices with one end leaning to a core vertex, respectively  $v_3R_2v_4Rr, v_3R_2v_4S_1s_1, v_3R_2v_4S_2s_2$ . If  $v_4$  does not lean to one of  $a_4$  or  $a_2$ , then  $R_2$  is either a skip or an  $M$ -alternating link. So we may assume  $v_4$  leans to one of  $a_4$  or  $a_2$  and by symmetry may assume  $a_4$ . But then rerouting  $Q_{a_1}$  as  $x_1Q_{a_1}v_1R_1v_2Q_{a_1}a_1$  gives a skip  $a_4R_2v_3Q_{a_1}uS_2x_2$  which gives a perfect weave.  $\square$

## 4.6 The Main Result

We now have the tools to prove the main result of this chapter:

**Theorem 4.6.1.** *A brace has a Pfaffian orientation if and only if it is isomorphic to the Heawood graph, or if it can be obtained from planar braces by repeated application of the trisum operation.*

We require the following easy results found as (6.4) and (6.5) in [42]:

**Theorem 4.6.2.** *Let  $G$  and  $H$  be bipartite graphs such that  $G$  contains  $H$  as a matching minor. If  $G$  has a Pfaffian orientation, then so does  $H$ .*

**Theorem 4.6.3.** *Let  $G_0$  be a graph,  $C$  a cycle in  $G$  with a perfect matching in the complement,  $G_1$  and  $G_2$  subgraphs of  $G_0$  such that  $G_1 \cap G_2 = C, G_1 \cup G_2 = G_0, V(G_1) \cap V(G_2) \neq \emptyset$ , and  $V(G_2) - V(G_1) \neq \emptyset$ . Let  $G$  be obtained from  $G_0$  by deleting some (possibly none) of the edges of  $C$ . Then if  $G_1$  and  $G_2$  have Pfaffian orientations, then so does  $G$ .*

Note that these immediately imply the following

**Theorem 4.6.4.** *Let  $G$  be graph that contains a perfect Rotunda-skein. Let  $\{x_1, x_2, x_3, x_4\}$  be the pivots of the Rotunda and suppose that  $G \setminus \{x_1, x_2, x_3, x_4\}$  has at least*



three components. Let  $G^+$  be the graph formed from  $G$  by adding the edges  $x_1x_2, x_2x_3, x_3x_4, x_4x_1$ . Let  $A_1, A_2, A_3$  be the three components of  $G \setminus \{x_1, x_2, x_3, x_4\}$  containing the three cores of the Rotunda-skein. Let  $A_i^+, i \in \{1, 2, 3\}$  be the subgraph of  $G^+$  induced on  $V(A_i) \cup \{x_1, x_2, x_3, x_4\}$ . Then  $G$  has a Pfaffian orientation if and only if each  $A_i^+$  has a Pfaffian orientation.

*Proof.* Suppose first that each  $A_i^+, i \in \{1, 2, 3\}$  has a Pfaffian orientation. Then we apply Theorem 4.6.3 with  $G_1 = A_1^+, G_2 = A_2^+, C = x_1x_2x_3x_4$  to see that  $A_1^+ \cup A_2^+$  has a Pfaffian orientation. Applying Theorem 4.6.3 once more with  $G_1 = A_1^+ \cup A_2^+, G_2 = A_3^+, C = x_1x_2x_3x_4$  and then deleting the edges of  $C$  not in  $G$  shows that  $G$  has a Pfaffian orientation.

In the other direction, we note that each of the graphs  $A_i^+$  is a matching minor of  $G$ , so the result follows immediately from Theorem 4.6.2.  $\square$

We are now prepared to prove Theorem 4.1.1

*Proof.* If  $G$  is the Heawood graph, then it contains a Pfaffian orientation (one is given in, for example, (6.3) [42]). If  $G$  is formed from repeated application of the trisum operation on graphs with Pfaffian orientations, then repeated application of Theorem 4.6.3 gives that  $G$  has a Pfaffian orientation as well. Since planar graphs have Pfaffian orientations, this direction is complete.

For the other direction, suppose  $G$  has a Pfaffian orientation. Let  $G$  be the counterexample with the fewest vertices. If  $G$  is planar, we are done. Otherwise, by Theorem 4.1.2,  $G$  contains either a perfect skein with pattern one of  $K_{3,3}$ , Rotunda, or the Heawood graph. By Little's Theorem, since  $G$  has a Pfaffian orientation, it does not contain a perfect  $K_{3,3}$  skein. If  $G$  contains a perfect Heawood-skein and no perfect  $K_{3,3}$ -skein, then by Theorem 4.5.3,  $G$  is isomorphic to the Heawood graph. So we may assume that  $G$  contains a perfect Rotunda-skein. Let  $\{x_1, x_2, x_3, x_4\}$  be the pivots of that Rotunda-skein. Then by Theorem 4.5.7,  $G \setminus \{x_1, x_2, x_3, x_4\}$  has at

least 3 components.

Let  $G^+$  be the graph formed from  $G$  by adding the edges  $x_1x_2, x_2x_3, x_3x_4, x_4x_1$ . Let  $A_1, A_2, A_3$  be the three components of  $G \setminus \{x_1, x_2, x_3, x_4\}$  containing the three cores of the Rotunda-skein. Let  $A_i^+, i \in \{1, 2, 3\}$  be the subgraph of  $G^+$  induced on  $V(A_i) \cup \{x_1, x_2, x_3, x_4\}$ . Then  $G$  is a trisum of  $A_1^+, A_2^+, A_3^+$  and, by Theorem 4.6.4, since  $G$  has a Pfaffian orientation, so do each of  $A_1^+, A_2^+, A_3^+$ . We need to show that each is a brace. Note that  $G^+$  is a brace. Fix  $i \in \{1, 2, 3\}$ . Then pick two edges in  $A_i^+$ ,  $e$  and  $f$ . Then  $G^+$  has a perfect matching containing  $e$  and  $f$ . Note that  $A_i^+$  has the same number of vertices of each parity. If none of  $\{x_1, x_2, x_3, x_4\}$  are matched to vertices outside of  $A_i^+$ , then we are done. If all four are, then we restrict the matching to  $A_i^+$  and add the edges  $x_1x_2, x_3x_4$ . Otherwise exactly two of opposite parity are matched to vertices outside of  $A_i^+$ , say  $x_1, x_2$ , so restrict the matching to  $A_i^+$  and add the edge  $x_1x_2$ .

Since each of  $A_i^+, i \in \{1, 2, 3\}$  is a brace and has fewer vertices than  $G$  and has a Pfaffian orientation, each is either isomorphic to the Heawood graph, or formed from planar braces by repeated application of the trisum operation by the minimality of  $G$ . Since the Heawood graph does not have any four-cycles, each of these graphs cannot be isomorphic to the Heawood graph, which completes the proof.  $\square$

## 4.7 An Algorithmic Consequence

The primary advantage to Theorem 4.1.2 in [42] over the previous results was that it provided a polynomial-time algorithm to determine whether or not a bipartite graph has a Pfaffian orientation. We note in this section that our results also provide such an algorithm.

We note first that the proofs in this section are fundamentally algorithmic; we make use of that fact to present the following sequence of theorems.

**Theorem 4.7.1.** *There exists a polynomial-time algorithm with input a non-planar*

brace,  $G$ , and output an odd hex contained in  $G$ .

*Proof.* This follows immediately from the proof of Theorem 3.1.3, noting that braces are internally 4-connected.  $\square$

**Theorem 4.7.2.** *There exists a polynomial-time algorithm with input a non-planar brace  $G$  and an odd hex contained in  $G$ ,  $K$ , and output a perfect  $H$ -skein in  $G$  where  $H$  is one of Rotunda, the Heawood graph or  $K_{3,3}$ .*

*Proof.* The proof of Theorem 4.4.4 is fundamentally algorithmic. Given an odd hex, we find a maximal matching in the complement which gives us a skein. As long as the matching in the skein is not perfect, if the pattern is Rotunda or Heawood, we find a snarl of width two and find a weave as in the proofs of Theorems 4.4.2 and 4.4.3. Otherwise, we find a snarl of width three and use that snarl to find a skein of larger heft as in the proof of Theorem 4.4.4. Since the heft of the skein increases in each iteration, there are at most a linear number of iterations, so this algorithm is polynomial-time.  $\square$

**Theorem 4.7.3.** *There exists a polynomial-time algorithm with input a brace  $G$  and output whether or not  $G$  has a Pfaffian orientation.*

*Proof.* We test  $G$  for planarity. If  $G$  is planar, then it has a Pfaffian orientation. Otherwise, we apply the algorithm of Theorem 4.7.1 to find an odd hex in  $G$ . We then apply the algorithm of Theorem 4.7.2. If we find a perfect weave, then  $G$  does not have a Pfaffian orientation. If the pattern of the skein is the Heawood graph, then either  $G$  is isomorphic to the Heawood graph in which case it has a Pfaffian orientation or, by Theorem 4.5.3, it contains a perfect weave in which case it does not have a Pfaffian orientation. Finally, if the pattern of the skein is Rotunda, we delete the four pivots of the Rotunda-skein and check whether  $G$  remains connected. If it does, then by Theorem 4.5.7,  $G$  contains a perfect weave. Otherwise, we recur

on each of the resulting components. If any of the components do not have a Pfaffian orientation, then  $G$  does not, and if all three do have Pfaffian orientations, then so does  $G$  by Theorem 4.6.4.  $\square$

We would now like to extend these results to bipartite graphs rather than only to braces. The following theorem follows immediately from (9.1) and (9.6) from [42], neither of which is difficult:

**Theorem 4.7.4.** *There exists a polynomial-time algorithm with input a bipartite graph  $G$  and output a list of braces,  $C_0, \dots, C_k$  such that  $G$  has a Pfaffian orientation if and only if each of  $C_0, \dots, C_k$  does.*

Combined with Theorem 4.7.3 this immediately gives the following:

**Theorem 4.7.5.** *There exists a polynomial-time algorithm with input a bipartite graph  $G$  and output whether or not  $G$  has a Pfaffian orientation.*

## CHAPTER V

### FLAT EMBEDDINGS

#### 5.1 *Introduction*

We are interested in this chapter in embeddings of graphs in 3-space. For their convenience, we remind the reader of several terms described in detail in Chapter 1. Every graph admits an embedding in  $S^3$ , called the book-embedding, so we require some additional restrictions to have a meaningful concept. One natural condition is to insist that every two disjoint cycles form a trivial link. This leads to the following definition: We say that an embedding of a graph  $G$  is *linkless* if for every two disjoint cycles of  $G$  their linking number is 0. Similarly, an embedded graph is *flat* if every cycle  $C$  of the graph bounds a disk  $\Delta$  disjoint from the rest of the graph. Such a disk is called a *panel* for  $C$ . We restate the following theorem of [41] from Chapter 1 that characterizes linkless and flat graphs:

**Theorem 5.1.1.** *For a graph  $G$  the following conditions are equivalent.*

- (1)  *$G$  has a flat embedding*
- (2)  *$G$  has a linkless embedding*
- (3)  *$G$  has no minor isomorphic to a member of the Petersen family*

In most of this section, we are generally more interested in  $Y - \Delta$  minors than in true minors. As discussed in the Introduction, let  $G$  be a graph and  $v \in V(G)$  of degree 3 with neighbors  $a, b, c$ . Then a  $Y - \Delta$  transformation is the removal of  $v$  and the addition of edges  $ab, ac$ , and  $bc$  to  $G$ . Similarly, let  $abc$  be a triangle in  $G$ . Then a  $\Delta - Y$  transformation is the removal of the edges  $ab, ac, bc$  and the addition of a vertex  $v$  adjacent to each of  $a, b, c$ . Further, if  $H$  is a graph, we say that  $G$  contains

$H$  as a  $Y - \Delta$  minor if there is a sequence of contractions,  $Y - \Delta$  operations, and  $\Delta - Y$  operations that take a subgraph of  $G$  to a graph isomorphic to  $H$ . Since  $Y - \Delta$  operations do not change whether or not a graph is flat, the above theorem implies that a graph contains a graph in the Petersen family as a proper minor if and only if it contains it as a  $Y - \Delta$  minor.

While an algorithm of van der Holst [50] can find a linkless embedding for a graph in polynomial time, there is no similar algorithm for flat embeddings. To that end, we are interested in the question of how to build up flat embeddings of graphs from flat embeddings of smaller graphs. One possibility is to glue two graphs together on small cutsets. In fact, the proofs of (5.3) and (5.4) of [41] give that

**Theorem 5.1.2.** *Let  $G$  be a graph with separation  $(A, B)$ ,  $K = A \cap B$ ,  $|K| \leq 4$ . Further if  $|K| = 4$ ,  $G \setminus K$  has exactly two components. Let  $K^*$  be the complete graph with vertex set  $K$  and  $\Gamma_A$  and  $\Gamma_B$  be flat embeddings for  $G[A] \cup K^*$  and  $G[B] \cup K^*$ . Then there is a natural flat embedding for  $G$  formed from  $\Gamma_A$  and  $\Gamma_B$ .*

We refer to separations of this sort as *complete separations*. Specifically, let  $(A, B)$  be a separation in a graph  $G$  with  $K = A \cap B$ ,  $K^*$  complete on  $K$ . Then the separation is complete if  $|K| \leq 4$ ,  $G \setminus K$  has exactly two components if  $|K| = 4$ , and  $G$  contains  $A \cup K^*$  and  $B \cup K^*$  as  $Y - \Delta$  minors. We note also that in order to create the embedding for  $G$  above, we require the panels for the embeddings of  $A$  and  $B$  on  $K^*$ .

A different possibility is to find an edge  $e$  such that  $G \setminus e$  has a flat embedding along with a natural way to add  $e$  back in. One way that we can do this is to find a *peripheral theta*:

**Definition.** Let  $C$  be a cycle with vertices  $v_1, \dots, v_n$  in order and let  $e = v_i v_j$  be an edge not in  $C$ . Then we refer to the graph with vertex set  $V(C)$  and edge set  $E(C) \cup \{e\}$  as an *theta graph* with arc  $e$  and loop  $C$ . A subgraph  $H$  in a graph  $G$  isomorphic to a theta graph is *peripheral* if  $H$  is induced in  $G$  and  $G - H$  is connected.

As will be shown later, if we have a flat embedding of a graph  $H$  found by deleting the arc of a peripheral theta with loop  $C$  from  $G$ , we can find a flat embedding for  $G$  by embedding  $e$  along a panel for  $C$ .

One nice aspect of flat embeddings is that, assuming the graph is reasonably well-connected, its embedding is unique. Specifically, we say that a *Kuratowski subgraph* is a subdivision of  $K_5$  or  $K_{3,3}$ . A graph is *Kuratowski-connected* if it is 3-connected and, for every separation  $(A, B)$  with  $|A \cap B| \leq 3$  there is a planar embedding of either  $G[A]$  or  $G[B]$  with  $A \cap B$  embedded on the outer face. The following theorem of [41] provides that.

**Theorem 5.1.3.** *Let  $G$  be a Kuratowski-connected graph that admits a flat embedding. Then all flat embeddings of  $G$  are homeomorphic.*

The main results of this paper are to show that, given a graph  $G$ ,  $G$  is either of bounded size, contains one of a Petersen family minor, a peripheral theta, or a complete separation, or is apex. By the above theorems, then, this gives us a strategy for an algorithm. If  $G$  is small, planar, or apex, embed it. If  $G$  contains a minor in the Petersen family, there is no possible flat embedding. If  $G$  contains a complete separation, recursively embed each of the sides of the separation and then glue them together using Theorem 5.1.2. Otherwise, we find a peripheral theta. Delete the arc of the theta, embed the resulting graph recursively, and then embed the arc across the panel for the loop of the theta.

There are two main difficulties with this algorithm. The first is that in addition to the embedding for the graph, we also require panels on select cycles in order to continue building up our embedding. It seems difficult to find such panels dynamically and unclear how to maintain such panels throughout the process. The second is in bounding the complexity of the embedding by a polynomial. If we measure the complexity of an embedding as the smallest number of crossings in regular projection, a naive implementation could easily cause the complexity to double (or more) each

time we embed the arc of some peripheral theta. In fact, it is not clear that, in general, flat graphs have a flat embedding that is of polynomial complexity in their number of vertices and edges. These difficulties seem a fruitful subject for further research and we hope that the theorems provided in this chapter are helpful in the eventual construction of such a polynomial time algorithm.

## 5.2 *The Main Theorem*

The following is the main theorem for this chapter:

**Theorem 5.2.1.** *There exists an absolute constant  $N$  such that if  $G$  is a graph on at least  $N$  vertices at least one of the following holds:*

- (1)  *$G$  contains a graph in the Petersen family as a minor*
- (2)  *$G$  contains a complete separation*
- (3)  *$G$  contains a peripheral theta, the deletion of whose arc leaves  $G$  Kuratowski-connected*
- (4) *There exists  $X \subseteq V(G)$ ,  $|X| \leq 1$  such that  $G \setminus X$  is planar*

As an applications to a flat embedding algorithm, the first and fourth outcomes here are obviously helpful. The second outcome is helpful as a result of Theorem 5.1.2. We discuss the third outcome in a little more detail. Specifically we would like to show that

**Theorem 5.2.2.** *Let  $G$  be a flatly embeddable graph. Assume that  $G$  has a peripheral theta graph with cycle  $C$  and arc  $e$ , and assume also that  $G \setminus e$  is Kuratowski connected. Let  $\phi$  be a flat embedding of  $G \setminus e$ , let  $\Delta$  be a panel for  $C$  in the embedding  $\phi$ , and let  $\psi$  be an embedding of  $G$  such that  $\psi(e) \subseteq \Delta$  and  $\psi(x) = \phi(x)$  for every vertex and edge  $x$  of  $G \setminus e$ . Then  $\psi$  is a flat embedding of  $G$ .*

We provide a proof here, but first require a similar lemma of [41]:



**Lemma 5.2.3.** *Let  $X \subseteq \mathbb{R}^3$  be closed, let  $F \subseteq X$  be a circle such that  $X - F$  is arc-wise connected and let  $y_1, y_2, y_3 \in F$  be distinct such that  $(X - F) \cup \{y_i\}$  is arc-wise connected for  $i = 1, 2, 3$ . Let  $\Delta_1, \Delta_2$  be  $X$ -panels for  $F$ . Then there is an orientation-preserving homeomorphism of  $\mathbb{R}^3$  fixing  $X$  and a neighborhood of  $X - F$  point wise and mapping  $\Delta_1$  to  $\Delta_2$ .*

We will also make use of Böhme's Lemma [6]:

**Lemma 5.2.4.** *Let  $C_1, \dots, C_n$  be cycles with pairwise connected intersection in an embedded graph  $G$ . If there is a panel for each of  $C_1, \dots, C_n$ , then there exist simultaneous panels  $\Delta_1, \dots, \Delta_n$  for  $C_1, \dots, C_n$  respectively so that  $\Delta_i \cap \Delta_j = C_i \cap C_j$  for  $1 \leq i < j \leq n$ .*

We now prove Theorem 5.2.2

*Proof.* Since  $G$  is a flatly embeddable graph, there exists a flat embedding of  $G$ ,  $\Gamma$ . Look at the restriction of  $\Gamma$  to  $G \setminus e$ ,  $\Gamma'$ . Then there is a homeomorphism between  $\Gamma'$  and  $\phi$  by Theorem 5.1.3 since  $G \setminus e$  is Kuratowski connected. We can choose this homeomorphism to be the identity by appropriately modifying  $\Gamma$ . Let  $\Delta$  be a panel for  $C$  in  $\phi$ . Let  $C_1$  and  $C_2$  be the two cycles in the theta graph using  $e$ . By Böhme's Lemma, we can find simultaneous panels for  $C_1$  and  $C_2$  in  $\Gamma$  whose union gives a panel for  $C$  containing  $e$ ,  $\Delta'$ . Note that since  $G$  is 3-connected and  $G \setminus C$  is connected, there exist three vertices on  $C$ ,  $v_1, v_2, v_3$ , such that  $G \setminus C \cup \{v_i\}$  is connected for  $1 \leq i \leq 3$ . By Lemma 5.2.3, there is a homeomorphism transforming  $\Delta$  into  $\Delta'$  which fixes the remainder of the embedding. But this has then provided a homeomorphism between  $\psi$  and  $\Gamma$ , so  $\psi$  is a flat embedding.  $\square$

As a final remark, we note that Theorem 5.2.1 only applies to sufficiently large graphs. But embedding graphs of bounded size can be done by the following algorithm:

**Theorem 5.2.5.** *Let  $G$  be a graph that admits a flat embedding. Then there is a finite time algorithm to find a flat embedding.*

*Proof.* For each integer  $i \geq 0$ , let  $H$  be formed from  $G$  by,  $i$  times, choosing a pair of edges of  $G$ ,  $e = uv$  and  $f = xy$ , removing them, and adding a new labeled vertex  $z$  and edges  $uz, vz, xz, yz$ . A planar embedding of  $H$  along with a choice of “up” or “down” for each labeled vertex gives a regular projection for an embedding of  $G$ . For each of these regular projections, check whether the corresponding embedding is flat by using the algorithm of [45]. If so, stop and return that embedding.

It is clear that, if the algorithm terminates, it returns a flat embedding for  $G$ .  $G$  by assumption admits a flat embedding. Therefore there exists a regular projection for that embedding with a finite number of crossing edges,  $n$ . Then this algorithm will encounter that regular projection when  $i = n$ , so will terminate.  $\square$

### 5.3 Structural Theorems

In this chapter we make use of several more sophisticated results from structural graph theory than are required in the previous chapter. For the convenience of the reader, we use this section to summarize and provide some context for these results.

We begin with the two-path theorem, a version of Theorems (2.3) and (2.4) in [36]:

**Theorem 5.3.1.** *Let  $G$  be a graph with  $x_1, x_2, y_1, y_2 \in V(G)$ . Further, suppose there is no separation  $(A, B)$  with  $|A \cap B| \leq 3$ ,  $\{x_1, x_2, y_1, y_2\} \subseteq A$  and  $B - A \neq \emptyset$ . Then either there exist two paths  $P_1, P_2$  with ends  $x_1, x_2$  and  $y_1, y_2$  respectively or there exists a planar embedding of  $G$  with  $x_1, y_1, x_2, y_2$  in that order along the boundary.*

The other major idea from graph minors theory that we make use of is tree-width and tree decompositions. Tree decompositions have a long history, first discovered in [18] and later rediscovered in [35] and [5]. A tree decomposition of a graph  $G$  is a

pair  $(T, Y)$  in which  $T$  is a tree and  $Y$  is a family of vertex sets indexed by vertices of the tree such that:

(P1)  $\bigcup_{t \in V(T)} Y_t = V(G)$ , and every edge of  $G$  has both ends in some  $Y_t$

(P2) If  $t, t', t'' \in V(T)$  and  $t'$  lies on the path between  $t$  and  $t''$  in  $T$ , then  $Y_t \cap Y_{t''} \subseteq Y_{t'}$

The *width* of a tree-decomposition is  $\max_{t \in V(T)} |Y_t - 1|$  and the *tree-width* of a graph is the minimum width of a tree-decomposition for it. The idea of tree-width has been studied extensively; we make use of several relevant theorems later and will state them as needed.

The final structural result we need comes from [39]. Suppose  $H$  is a minor of  $G$ . Then there is a partition of a subset of the vertex set of  $G$  into connected subgraphs  $A_1, A_2, A_3, \dots, A_k$  such that if  $v_1, v_2, \dots, v_k$  are the vertices of  $H$  and there is an edge between  $v_i$  and  $v_j$  in  $H$ , there is an edge between  $A_i$  and  $A_j$  in  $G$ . We refer to the  $A_i$  as the *bags* of the minor.

Let  $\{a, b, c, d\}$  be four distinct vertices in a graph  $G$ . Then we say that an  $\{a, b, c, d\}$ -minor is a  $K_4$  minor of  $G$  in which each of  $\{a, b, c, d\}$  are in separate bags of the minor. The following is an immediate consequence of (2.6) from [39]:

**Theorem 5.3.2.** *Let  $G$  be a graph with  $a, b, c, d \in V(G)$  and  $|V(G)| \geq 6$ . Further, suppose there is no separation  $(A, B)$  with  $|A \cap B| \leq 3$ ,  $\{a, b, c, d\} \subseteq A$  and  $B - A \neq \emptyset$ . Then one of the following holds:*

- (1)  $G$  has an  $\{a, b, c, d\}$  minor
- (2) There is a planar embedding of  $G$  with  $a, b, c, d$  on the outer face
- (3) There exist vertices  $x$  and  $y \in V(G) - \{a, b, c, d\}$  such that  $G \setminus \{x, y\}$  has three components, two of which are exactly single vertices of  $\{a, b, c, d\}$

## 5.4 Connectivity

In this section we detail a theorem that allows us to handle sufficiently large graphs that are not well-connected. We say that a graph is *internally 5-connected* if it is 4-connected and, for every separation  $(A, B)$ ,  $|A \cap B| = 4$ , either  $|A| \leq 5$  or  $|B| \leq 5$ .

Specifically, we show that

**Theorem 5.4.1.** *Let  $G$  be a 4-connected graph. Let  $A, B$  be a 4-separation of  $G$  with  $|A \cap B| = X$ ,  $|V(B)| \geq 6$ ,  $|V(A)| \geq 5$ . Then either  $G$  contains  $K_{4,4}$  minus an edge as a  $Y - \Delta$  minor,  $G$  contains a peripheral theta, or both  $A$  and  $B$  contain  $K_4$  minors rooted on  $X$  and are connected after deleting  $X$ .*

Several of the definitions required for this theorem will follow later in the section, but for the moment we note that this theorem is useful in reducing the problem of finding a flat embedding to smaller graphs. Without definition, if  $G$  contains  $K_{4,4}$  minus an edge as a  $Y - \Delta$  minor, then it is not flat. If  $G$  contains a peripheral theta, that is the same result we are finding from our main theorem in the case of internal 5-connectivity. Finally, if both  $A$  and  $B$  contain rooted  $K_4$  minors, again without definition, a theorem of [41] will allow us to glue together a flat embedding found on the two sides.

We begin by handling the case that  $G$  contains a large planar piece in a particular way.

**Lemma 5.4.2.** *Let  $G$  be a 3-connected graph. Let  $H$  be an induced plane subgraph of  $G$  with outer cycle  $C$  and  $|E(H)| > |E(C)|$ . Suppose that  $G - H$  is connected and that for any set  $X \subseteq V(H)$  with  $|X| \leq 3$ , no component of  $V(H) - X$  is disjoint from  $C$ . Then  $G$  contains a peripheral theta which is a subgraph of  $H$ .*

*Proof.* We proceed by induction on  $|V(H)| + |E(H)|$ . Clearly if  $|V(H)| = 4$  and  $|E(H)| = 5$ , then  $H$  is itself a peripheral theta in  $G$ . Note that  $H$  is 2-connected, so the boundary of every face is a cycle.

Suppose  $H$  contains an edge  $e$  such that the bounding cycles of the two faces incident with  $e$  are completely disjoint from  $C$ . Let  $u$  and  $v$  be the ends of  $e$  and  $F_1$  and  $F_2$  the bounding cycles of the incident faces. Then we claim that  $F = F_1 \cup F_2$  is a peripheral theta of  $G$ . Note first that  $F_1$  and  $F_2$  intersect in exactly  $e$  since if they met in another vertex  $w$ , then either  $u$  and  $w$  separate  $v$  from  $C$  or  $v$  and  $w$  separate  $u$  from  $C$ . So the vertices of  $F_1 \cup F_2$  form a cycle, which we label  $v = v_1, v_2, \dots, u = v_n, v_{n+1}, \dots, v_m$ . Suppose  $F$  is not induced. Then  $f = v_i v_j$  is an edge,  $i < j$ , but then the set  $\{v_i, v_j, u\}$  or  $\{v_i, v_j, v\}$  separates part of  $F$  from  $C$ . Similarly, let  $S$  be a component of  $G - F$  disjoint from  $G - H$ . Then  $S$  is disjoint from  $C$ . Let  $i$  be the smallest number such that  $v_i$  is a neighbor of a vertex in  $S$  and  $j$  the largest. Then  $\{v, v_i, v_j\}$  separates  $S$  from  $C$  which is a contradiction. So  $F$  is a peripheral theta.

Let  $V(C) = \{v_1, v_2, v_3, \dots, v_n\}$  in order. Suppose  $H$  contains an edge  $e$  with ends  $v_i, v_j$ ,  $n - 1 > j - i > 1$ . If  $H$  is exactly  $C \cup \{e\}$ , then  $H$  is a peripheral theta in  $G$ . Otherwise, let  $H'$  be the plane subgraph of  $H$  bounded by  $v_i, v_{i+1}, \dots, v_j$  if it has at least one edge not in  $C$  and the plane subgraph of  $H$  bounded by  $v_j, v_{j+1}, \dots, v_i$  otherwise. Let  $C'$  be the outer cycle of  $H'$ . Note that since  $C \cup \{e\}$  was not all of  $H$ ,  $H'$  satisfies  $|E(H')| > |E(C')|$ .  $H'$  is clearly induced since  $H$  is and  $G - H'$  is connected since  $\{v_i, v_j\}$  does not separate  $G$  since  $G$  is 3-connected. So by induction  $G$  contains a peripheral theta which is a subgraph of  $H'$  and hence of  $H$ .

So we may assume that  $H$  contains a vertex not on  $C$ ,  $v$ . By Menger's Theorem, there are 4-vertex disjoint paths, each with one end  $v$  and the other end on  $C$ . Let  $P_1, \dots, P_4$  be these paths and let the ends of  $P_i$  be  $v$  and  $u_i$  where  $u_1, u_2, u_3, u_4$  are in order around  $C$ . Further, choose these paths with total length as small as possible. Since  $G$  is 3-connected,  $G - H$  has a neighbor in one of the paths  $u_1 C u_2 C u_3$  and  $u_1 C u_4 C u_3$  other than  $u_1$  and  $u_3$ . Without loss of generality, we may assume  $G - H$  has a neighbor in  $u_1 C u_2 C u_3$  other than  $u_1$  or  $u_3$ . Let  $H'$  be the plane subgraph of  $H$

bounded by  $u_1P_1vP_3u_3Cu_4Cu_1$ . Let  $C'$  be the outer cycle of  $H'$ . Then  $H'$  is induced since the  $P_i$  are as short as possible. Since  $P_4 \subset H'$ ,  $|E(H')| > |E(C')|$ . Finally,  $C - C'$  and  $G - H$  are in the same component of  $G - H'$  by our choice of  $H'$ . Suppose there is another component  $S$  in  $G - H'$ . Then  $S$  has 4 neighbors on exactly one of  $P_1$  or  $P_3$ , so we may assume on  $P_1$ . Let these neighbors be  $x_1, x_2, x_3, x_4$  in order on  $P_1$  with  $x_1$  the closest to  $u_1$  along  $P_1$ . Then look at the two faces containing the edge from  $S$  to  $x_3$ . All of the vertices of their bounding faces must be disjoint from  $C$ , so we are done. So we may assume no such component exists, so we may apply induction to  $H'$  which completes the proof.

□

**Definition.** Let  $G$  and  $H$  be graphs with  $|V(H)| = n$ . An  $H$ -minor of  $G$  rooted at  $\{v_1, v_2, \dots, v_n\} \in V(G)$  is an  $H$ -minor of  $G$  in which each of the  $v_i$  are in separate bags of the minor.

**Lemma 5.4.3.** *Let  $G$  be a 4-connected graph. Let  $H$  be a plane subgraph of  $G$  with vertices  $a, b, c, d$  embedded on the outside in order. Further, suppose that  $H - \{a, b, c, d\}$  is a component of  $G - \{a, b, c, d\}$  and that  $|V(H)| \geq 6$ . Then either  $G$  contains a peripheral theta whose arc is disjoint from  $\{a, b, c, d\}$  or the subgraph of  $G$  induced by the vertices of  $H$  contains a  $K_4$  minor rooted at  $\{a, b, c, d\}$ .*

*Proof.* Let  $X = \{a, b, c, d\}$ . If  $ab, bc, cd, ad$  are edges in  $G - H$  add them to  $H$ .

Suppose  $H$  contains an edge  $e$  with ends not in  $X$  and whose incident faces are bounded by cycles. Let  $u$  and  $v$  be the ends of  $e$  and  $F_1$  and  $F_2$  the bounding cycles of the incident faces. Then we are interested in when  $F = F_1 \cup F_2$  is a peripheral theta of  $G$ . Note first that  $F_1$  and  $F_2$  intersect in exactly  $e$  since if they met in another vertex  $w$ , then either  $u$  and  $w$  separates  $v$  from  $X$  or  $v$  and  $w$  separates  $u$  from  $X$ . So the vertices of  $F_1 \cup F_2$  form a cycle, which we label  $v = v_1, v_2, \dots, u = v_n, v_{n+1}, \dots, v_m$ . Let  $S$  be a component of  $G - F$  disjoint from  $G - H$ . Then  $S$  is disjoint from  $X$ .

Let  $i$  be the smallest number such that  $v_i$  is a neighbor of a vertex in  $S$  and  $j$  the largest. Then  $\{v, v_i, v_j\}$  separates  $S$  from  $X$  which is a contradiction. Suppose  $F$  is not induced. Then  $f = v_i v_j$  is an edge,  $i < j$ , but then the set  $\{v_i, v_j, u\}$  or  $\{v_i, v_j, v\}$  separates part of  $F$  from  $C$  unless  $f$  is not contained in  $H$ , so is between two vertices of  $X$ . So  $F$  is a peripheral theta unless it contains two vertices of  $X$ , with an edge between them in  $G - H$ .

Let  $v \in V(H) - \{a, b, c, d\}$ . Then by Menger's Theorem,  $v$  has vertex disjoint paths  $P_1, P_2, P_3, P_4$  to  $a, b, c, d$  respectively. Choose these so that their total length is as short as possible. Suppose each of these paths has length 1. By assumption,  $H$  contains another vertex  $u$  which must, in the planar embedding, lie in one of the four quadrants given by the  $P_i$ . We may assume it lies in the quadrant formed by  $P_1$  and  $P_2$ . But then  $\{v, a, b\}$  is a separation in  $G$  which is a contradiction. So at least one of these paths is not a single edge. We assume that path is  $P_1$  and  $u$  is the neighbor of  $v$  along  $P_1$ . Then by Menger's Theorem,  $u$  has vertex disjoint paths  $Q_1$  and  $Q_2$  to, respectively,  $x, y \in V(P_2 \cup P_4)$  that avoid  $a$  and  $v$ . If  $x \in V(P_2), y \in V(P_4)$ , then the faces incident with  $e = uv$  are bounded by cycles. Similarly, if  $x, y \in V(P_2)$  with  $x \in V(vP_2y)$ , then any edge along  $Q_1$  has ends disjoint from  $\{a, b, c, d\}$  and its neighboring faces are bounded by cycles. In this case, at most one vertex of  $\{a, b, c, d\}$ ,  $b$ , is contained in the bounding cycles of these faces, so by the above we are done.

So we may assume that the faces incident with  $e = uv$  are bounded by cycles  $F_1, F_2$ . By the above,  $F = F_1 \cup F_2$  is a peripheral theta unless it contains two vertices of  $X$  with an edge between them in  $G - H$ . Since  $c$  cannot be in  $F$ , those two vertices of  $X$  must be  $b$  and  $d$ . But then contract  $uP_1a$  and  $vP_3c$ ,  $P_2 - \{v\}$ , and  $P_4 - \{v\}$  to find the desired  $K_4$ -minor.

□

**Lemma 5.4.4.** *Let  $G$  be a 4-connected graph. Let  $A, B$  be a 4-separation of  $G$  with*

$|A \cap B| = X$ ,  $|V(A)|, |V(B)| \geq 6$ . Suppose that  $A$  does not contain a  $K_4$  minor rooted at  $X$  or a  $K_{4,2}$  minor with all the vertices of  $X$  in different components of the 4-side of the minor. Then  $G[A]$  contains a peripheral theta whose arc is disjoint from  $\{a, b, c, d\}$ .

*Proof.* Let  $a, b, c, d$  be the four vertices of the separation. We proceed by induction on the size of  $A$ . Suppose first that  $|A| = 6$ . Let  $x$  and  $y$  be the two other vertices of  $A$ . If  $x$  and  $y$  are not adjacent, then  $A$  is  $K_{4,2}$  with  $a, b, c, d$  the vertices on the 4-side. So we may assume  $x$  and  $y$  are adjacent. Since  $x$  and  $y$  are each degree at least 4, each is adjacent to at least three of  $a, b, c, d$ . So if any two of  $a, b, c, d$  do not neighbor both  $x$  and  $y$ , we can embed  $A$  planarly with  $a, b, c, d$  on the outside in which case we find a peripheral theta by Lemma 5.4.3. So exactly one of  $a, b, c, d$  neighbors only one of  $x$  and  $y$ , say  $a$  neighbors  $x$ . But then consider the peripheral theta with loop  $C = xbyd$  and arc  $xy$ .  $G$  remains connected after deleting  $V(C)$  since  $\{b, d\}$  does not disconnect  $G$ . Further,  $C$  is induced (other than  $xy$ ) unless  $bd$  is an edge. But then contracting  $a$  with  $x$  and  $c$  with  $y$  gives a rooted  $K_4$  minor in  $A$ .

Suppose  $|A| = 7$ . Let  $x, y, z$  be the vertices of  $A$  other than  $a, b, c, d$ . Then we note that we may assume  $xy$  and  $yz$  are edges since otherwise we are in the previous case in which  $|A| = 6$ . Further, each of  $a, b, c, d$  has at least 2 neighbors in  $x, y, z$  since otherwise we can apply induction and immediately find one of the desired outcomes. Suppose one of  $a, b, c, d$ , say  $a$  is adjacent to  $x, y$ , and  $z$ . We know that  $y$  has another neighbor in  $b, c, d$ , say  $b$ . Then we may assume  $x$  is adjacent to  $c$  and  $z$  to  $d$ . If  $x$  is also adjacent to  $d$ , contract  $xc, yb, zd$  to find a rooted  $K_4$  minor. So we have  $y$  adjacent to  $c$  and  $d$ . Since  $b$  has two neighbors in  $x, y, z$ , we may assume  $x$  neighbors  $b$ . Then  $z$  has degree at least 4, so neighbors either  $b$  or  $x$ . In either case, contracting  $xb, yc, zd$  gives the desired  $K_4$  minor. So we may assume none of  $a, b, c, d$  is adjacent to all of  $x, y, z$ . If  $xz$  is not an edge, then it is immediately clear that (up to symmetry), the edges are  $ax, az, bx, bz, cx, cy, dy, dz$ , in which case  $C = bxcyz$  with arc  $xy$  is a



peripheral theta since deleting  $C$  leaves  $G$  connected and it is induced unless  $cb$  is an edge, in which case contracting  $ax, bz, dy$  gives the rooted  $K_4$  minor. So we may assume  $xz$  is an edge. We may assume  $a$  neighbors  $x$  and  $y$ . But then  $C = axyz$  with arc  $xy$  is a peripheral theta since  $a$  and  $z$  are non-adjacent. So  $|A| > 7$ .

Suppose now that  $|A| \geq 8$ . Then by Theorem 5.3.2, either  $G$  has a rooted  $K_4$ -minor, a planar embedding with  $a, b, c, d$  on the outside, or two vertices  $x, y$  such that  $A \setminus \{x, y\}$  has two components, each of which a single vertex of  $a, b, c, d$ . We may assume the latter by Lemma 5.4.3. Suppose  $a$  neighboring  $x$  and  $b$  neighboring  $y$  are the single components after deleting  $x$  and  $y$ . But then  $x, y, c, d$  forms a four-separation that satisfies the induction hypothesis, so has either one of the desired minors or a peripheral theta. But a desired minor in  $x, y, c, d$  is also a desired minor in  $a, b, c, d$  after contracting  $ax$  and  $by$  which completes the proof.  $\square$

The graph  $K_4^+$  is the graph on 5 vertices,  $v_1, v_2, v_3, v_4, v_5$  such that  $v_5$  is adjacent to each of  $v_1, \dots, v_4$  and  $v_1v_2v_3$  forms a triangle. When we say that  $G$  contains a *rooted*  $K_4^+$  subgraph on vertices  $a, b, c, d$ , we mean that (up to permutation of  $a, b, c, d$ ),  $G$  contains a subgraph isomorphic to  $K_4^+$  in which  $a = v_1, b = v_2, c = v_3, d = v_4$ .

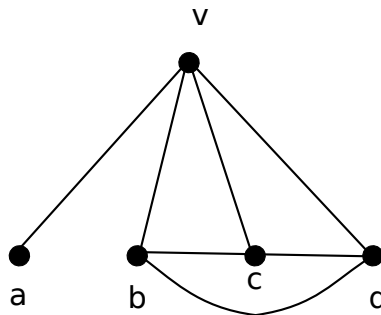


Figure 5.1: The graph  $K_4^+$

**Definition.** Let  $G$  be a graph and  $X = \{a, b, c, d\} \subseteq V(G)$ . Then we say that  $G$  contains  $K_4^+$  as a  $Y - \Delta$  minor rooted at  $X$  if there are a sequence of contractions,  $Y - \Delta$  operations, and  $\Delta - Y$  operations that take a subgraph of  $G$  to a graph with

a vertex  $v \notin X$  such that  $v$  is adjacent to each member of  $X$  and there are three vertices in  $X$ , say  $b, c, d$  such that  $bcd$  forms a triangle.

**Lemma 5.4.5.** *Let  $G$  be a graph on at least 5 vertices and suppose that  $X \subseteq V(G)$ ,  $|X| = 4$  such that  $G$  contains  $K_4$  as a  $Y - \Delta$  minor rooted at  $X$ . Further, suppose that there is no subset of  $X$ ,  $X'$ , such that  $G - X'$  is not connected. Then  $G$  contains  $K_4^+$  rooted at  $X$  as a  $Y - \Delta$  minor.*

*Proof.* We proceed by induction on  $|V(G)| + |E(G)|$ . If  $|V(G)| = 5$ , we have  $K_4^+$  as a rooted subgraph which suffices.

Let  $X = \{a, b, c, d\}$  and let the components of the rooted  $K_4$  minor be  $A, B, C, D$  respectively. Note first that each component of  $A - a$  must be a single vertex, since otherwise we contract it down to a single vertex and apply induction, and similarly for  $B - b$  and so on. In addition, each vertex of  $G - X$  has a neighbor in  $G - X$ , since  $|V(G - X)| \geq 2$  and  $G - X$  is connected. Next, we may assume each vertex of  $G - X$  has degree at least 4. If a vertex  $v$  had degree 1, we can delete it, 2, we can contract an incident edge, and 3, remove it with a  $Y - \Delta$  transformation, each of which reduces the number of vertices by 1.

For every vertex  $v$  of  $A - a$ , there is some member of  $\{B, C, D\}$ , say  $B$ , such that  $A - v$  has no neighbor in  $B$ . Otherwise, contract the edge between  $v$  and one of its neighbors in  $G - X$ . Then the connectivity requirements remain satisfied, as does the rooted minor requirement. In addition, for every vertex  $v$  of  $A - a$ , if  $v$  has a neighbor in  $B - b$ , then it is not adjacent to  $b$ , since otherwise we could just delete the edge to  $b$ .

If  $v \in A - a$  has a neighbor in each of  $B, C, D$ , then contract  $B, C, D$  to single vertices to find a rooted  $K_4^+$  as desired. So we may assume that  $v$  has 2 neighbors in  $B$ ,  $x$  and  $y$ . Since  $x$  and  $y$  form unique connections for  $B$ ,  $x$  has a neighbor in  $C$  and  $y$  in  $D$ . Note that  $v$  does not have another neighbor in  $B$ , so we may assume  $v$  has a neighbor in  $C$ . Further note that  $C$  and  $D$  are adjacent. Then use the  $Y - \Delta$

operation to remove  $y$  and similarly for  $x$ . Then contracting  $A - v$ ,  $B$ ,  $C$ ,  $D$  gives  $K_4^+$  as a rooted minor.  $\square$

The following lemma will prove useful in the proof of Theorem 5.4.7:

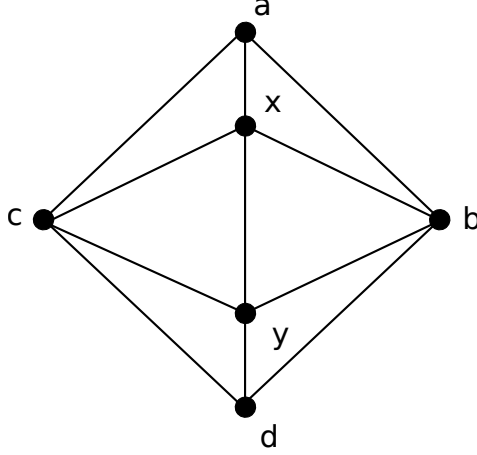


Figure 5.2: The embedding for Lemma 5.4.6

**Lemma 5.4.6.** *Let  $G$  be a graph and let  $a, b, c, d, x, y \in V(G)$  be distinct. Suppose there does not exist a separation  $(A, B)$  with  $\{a, b, c, d\} \subseteq A$ ,  $B - A \neq \emptyset$ , and  $|A \cap B| \leq 3$  and that  $G$  does not have a  $K_4$  minor rooted on  $\{a, b, c, d\}$ . Suppose  $e = xy \in E(G)$  and  $G \setminus \{b, c\} \setminus \{e\}$  has at least two components,  $U$  containing  $x$  and  $a$  and  $W$  containing  $y$  and  $d$ . Then there is a planar embedding of  $G$  with  $a, b, c, d$  on the outer face.*

*Proof.* Choose  $G$  to be a counterexample with  $|V(G)|$  as small as possible. Note that if  $|V(G)| = 6$ , such an embedding is easy to find by embedding  $abdc$  on the outer face in that order and embedding  $x$  and  $y$  inside as seen in Figure 5.2. Note that  $bc$  is not an edge since otherwise contracting  $x$  with  $a$  and  $y$  with  $d$  would give the rooted minor.

Suppose both  $a$  and  $d$  have degree 1. If the neighbor of  $a$  is not  $x$ , then deleting  $a$  and replacing it by its neighbor  $a'$  gives a smaller graph that satisfies the conditions of the theorem, so can be embedded with  $a', b, c, d$  on the outer face, so with  $a, b, c, d$

on the outer face. So the neighbor of  $a$  is  $x$  and of  $d$  is  $y$  for the same reasons. But  $G$  contains another vertex  $p$ , say in  $U$ , in which case deleting  $x, b, c$  separates  $p$  from  $a, b, c, d$ . So not both  $a$  and  $d$  have degree 1.

We apply Theorem 5.3.2. Since outcome (1) of that theorem is impossible and outcome (2) of the theorem is our desired result, we may assume that we have outcome (3). So there exist vertices  $p, q \in V(G) - \{a, b, c, d\}$  such that  $G \setminus \{p, q\}$  has three components, two of which are exactly single vertices of  $\{a, b, c, d\}$ . Note that there is no vertex in  $V(G) - \{a, b, c, d\}$  that is adjacent to both  $a$  and  $d$  and not both have degree 1, so  $a$  and  $d$  cannot be the two single vertex components. So we may assume that  $b$  is one of the single vertex components. If  $p$  and  $q$  are both in  $U$ , then deleting  $x$  and  $c$  separates  $y$  from  $a, b, c, d$ . So we may assume  $p \in U, q \in W$ . If  $c$  is the other single component, then deleting  $x, q, d$  separates  $W$  from  $a, b, c, d$  and deleting  $y, p, a$  separates  $U$  from  $a, b, c, d$ . Since one of  $U$  or  $W$  has at least three vertices, this is impossible.

So we may assume that  $a$  is the other single component. Since  $a$  does not neighbor  $q$ ,  $a$  has degree 1, so by the minimality of  $G$ ,  $p = x$ . But then  $U = \{x, a\}$  since otherwise deleting  $x, c$  separates  $U$  from  $a, b, c, d$ . Suppose  $y = q$ . Then deleting  $y, c, d$  separates  $W$  from  $a, b, c, d$  and since  $|W| \geq 3$ , this is a contradiction. So  $y \neq q$ . Then delete  $b$ , add the edge  $xq$  and replace  $b$  by  $q$  in the statement of the theorem. By the minimality of  $G$ , the resulting graph embeds with  $a, q, c, d$  on the outer face. Note that  $a$  has degree at most 2 and neighbors  $x$ , so  $x$  is on the outer face as well, so the  $xq$  edge is on the outer face. Replacing the  $xq$  edge with the path  $xbq$  then gives the desired embedding.

□

We are now prepared to prove the main theorem of this section.

**Theorem 5.4.7.** *Let  $G$  be a 4-connected graph. Let  $X, Y$  be a 4-separation of  $G$  with  $|X|, |Y| \geq 6$ . Then either  $G$  contains  $K_{4,4}$  minus an edge as a  $Y - \Delta$  minor,  $G$*

contains a peripheral theta with arc  $e$  such that  $G \setminus \{e\}$  is Kuratowski-connected, or  $G$  contains a separation  $(X', Y')$  of size 4,  $|X'|, |Y'| \geq 6$  such that both  $X'$  and  $Y'$  contain rooted  $K_4$  minors on  $X' \cap Y'$  and are connected after deleting  $X' \cap Y'$ .

*Proof.* Let  $(A, A')$  and  $(B, B')$  be separations of  $G$  with  $A \subseteq B'$  and  $B \subseteq A'$  such that  $|A \cap A'| = |B \cap B'| = 4$  and  $|A|, |B| \geq 6$ . Choose these sets so that as many of  $G[A]$  and  $G[B]$  as possible do not admit a planar embedding with, respectively,  $A \cap A'$  and  $B \cap B'$  on the outer face and, subject to that,  $|A| + |B|$  is as small as possible. Note that a choice of  $A, B, A', B'$  exists by taking  $A = B' = X, B = A' = Y$ .

We will refer to a peripheral theta with arc  $e$  such that  $G \setminus e$  is Kuratowski-connected as a *good peripheral theta*. We start by showing that one of  $A$  or  $B$  contains a peripheral theta and then find a good peripheral theta from there.

Suppose first that both  $A$  and  $B$  contain rooted  $K_4$  minors on, respectively  $A \cap A'$  and  $B \cap B'$ . Note that by Menger's theorem we can find four vertex disjoint paths between  $A \cap A'$  and  $B \cap B'$ , so we can find rooted  $K_4$  minors on  $A \cap A'$  in both  $A$  and  $A'$ . Suppose that either  $A - A'$  or  $A' - A$  is not connected. We may assume  $A - A'$  is not connected since the proof of the other case is identical. Then let  $S$  be a subgraph of  $A'$  that contains the rooted  $K_4$  minor. Then  $A$  contains a  $K_{4,2}$  minor with all the vertices in  $A \cap A'$  in distinct bags on the four side of the minor since each component of  $A - A \cap A'$  neighbors all four vertices in  $A \cap A'$ , so by Lemma 5.4.5 and the  $K_4$  minor in  $S$ ,  $G$  contains  $K_{4,4}$  minus an edge as a  $Y - \Delta$  minor.

If instead  $A - A'$  and  $A' - A$  are connected, we have a desired outcome, so we may assume  $B$  does not contain a  $K_4$  minor rooted on  $B \cap B'$ . Suppose that there is a  $K_4$  minor in  $A$  rooted in  $A \cap A'$ . Then we apply Lemma 5.4.4 to the separation  $(B, B')$  to find that  $B$  contains either a rooted  $K_{4,2}$  minor which, when combined with the  $K_4$  minor gives a  $K_{4,4}$  minus an edge  $Y - \Delta$  minor by Lemma 5.4.5, or  $B$  contains a peripheral theta.

Suppose instead that neither  $A$  nor  $B$  contains a rooted  $K_4$  minor. Then by

Lemma 5.4.4, each contains either a rooted  $K_{4,2}$  minor which gives a  $K_{4,4}$  minor when combined or at least one of the two contains a peripheral theta.

So we may assume that  $A$  does not contain  $K_4$  minor rooted at  $A \cap A'$  and does contain a peripheral theta with arc  $e = xy$ . Let  $A \cap A' = \{a, b, c, d\}$ . Note that Lemma 5.4.4 assures us that  $x, y \notin \{a, b, c, d\}$ . Suppose that  $G' = G \setminus \{e\}$  is not Kuratowski-connected. Then there is a set  $\{r, s, t\}$  such that  $G' \setminus \{e\}$  has a separation  $(U, V)$  with  $U \cap V = \{r, s, t\}$  and  $G'[U]$  and  $G'[V]$  each cannot be embedded in the plane with  $\{r, s, t\}$  on the outer face. Choose  $\{r, s, t\}$  subject to the above so that  $\{r, s, t\} \cap \{a, b, c, d\}$  is as large as possible.

Since  $G$  is 4-connected, we may assume  $x \in U, y \in V$ . We note that  $\{a, b, c, d\}$  are not necessarily distinct from  $\{r, s, t\}$ , but all these vertices must be distinct from  $x$  and  $y$ . Let  $Q_1 = A \cap U - \{a, b, c, d, s, r, t\}, Q_2 = A \cap V - \{a, b, c, d, s, r, t\}, Q_3 = (A') \cap U - \{a, b, c, d, s, r, t\}, Q_4 = (A') \cap V - \{a, b, c, d, s, r, t\}$ .

Suppose first that  $\{a, b, c, d\} \subseteq V$ . Then if  $Q_3$  is not empty,  $\{r, s, t\}$  is a 3-cut in  $G$ . Since  $G$  does not have a separation of size less than 4, we must have that all of  $\{r, s, t\}$  are in  $A$  since otherwise  $x$  and  $y$  are on either side of such a separation. By assumption,  $G[Q_1 \cup \{r, s, t\}]$  cannot be embedded in the plane with  $r, s, t$  on the outer face, so has at least 5 vertices. But then we choose a new  $A^* = Q_1 \cup \{y, r, s, t\}$  which is a strict subgraph of  $A$  and  $G[A^*]$  cannot be embedded in the plane with  $y, r, s, t$  on the outer face. Thus we may assume  $\{a, b, c, d\} \not\subseteq V$  and, by symmetry,  $\{a, b, c, d\} \not\subseteq U$ . Note that this implies that  $\{r, s, t\} \not\subseteq \{a, b, c, d\}$ .

Suppose next that two of  $\{r, s, t\}$  are elements of  $\{a, b, c, d\}$ . Then we may assume  $a = r, b = s$  and  $c \in U, d \in V$ . If  $t \notin A'$ , then since  $|B| \geq 6$ , at least one of  $\{c, r, s\}$  or  $\{r, s, d\}$  is a 3-cut, so we may assume  $t \in A'$ . By Lemma 5.4.6,  $G[A]$  can be embedded in the plane with  $a, b, c, d$  on the outer face. Since  $|B| \geq 6$ , we may assume that  $Q_3$  is not empty. Then take  $A^* = Q_1 \cup Q_3 \cup \{a, b, c, t\}$  and  $B^* = Q_2 \cup Q_4 \cup \{x, a, b, d, t\}$  to find sets  $A^*$  and  $B^*$  that satisfy the conditions for  $A$  and  $B$ , but do not allow an

embedding with  $A^* \cap B^*$  on the outer face.

Suppose next that three of  $\{a, b, c, d\}$  are in  $U$ . Then we may assume  $a, b \in U - V$  and  $c \in U$ . Since  $G$  does not contain a separation of size less than 4, at least two of  $s, r, t$  must be contained in  $A$ , say  $s$  and  $r$  (with  $r$  possibly the same as  $c$ ) since otherwise  $x$  and  $y$  would be on either side of such a separation. Since  $|B| \geq 6$ , at least one of  $Q_3$  or  $Q_4$  is non-empty. If  $Q_4$  is non-empty, then there is a small cut containing  $d$  and up to two of  $c, t$  that separates it from the rest of the graph. So  $Q_4$  is empty, so  $Q_3$  is non-empty, so, since  $a, b, c$  is not a cut,  $t \in A' - A$ . Note that  $d$  and  $t$  must be neighbors since  $t$  has vertex disjoint paths to each of  $a, b, c, d$ . Let  $S_1 = Q_1 \cup Q_3 \cup \{a, b, c, s, r, t\}$  and  $S_2 = Q_2 \cup \{s, r, t, d\}$ . Then  $(S_1, S_2)$  is a separation in  $G' = G - \{xy\}$  with intersection  $\{s, r, d\}$ . If  $G'[S_1]$  could be embedded in the plane with  $\{s, r, d\}$  on the outer face, then  $G'[S_1 \cup \{t\}]$  can be embedded in the plane with  $\{s, r, t\}$  on the outer face since  $t$  neighbors at most two of  $\{s, r, d\}$ . Similarly, if  $G'[S_2]$  can be embedded in the plane with  $s, r, d$  on the outer face then  $G'[S_2 - \{d\}]$  can as well with  $\{s, r, t\}$  on the outer face since  $d$  and  $t$  are neighbors. But then  $\{s, r, d\}$  contradicts the choice of  $\{s, r, t\}$  since it has more overlap with  $\{a, b, c, d\}$ .

So we must have that all of  $a, b, c, d, r, s, t$  are distinct and  $\{a, b\} \subset U, \{c, d\} \subset V$ . One of  $Q_3$  or  $Q_4$  is non-empty, so we must have two of  $\{r, s, t\}$  in  $A'$ , say  $s, t$ . Similarly,  $\{x, c, d\}$  is not a cut, so we must have  $r \in A$ . But then  $xy$  could not be the arc of a peripheral theta contained entirely in  $A$ , since that would require three vertex disjoint paths between  $x$  and  $y$  in  $A$ , but the  $xy$  edge and  $r$  separate  $x$  from  $y$ .  $\square$

## 5.5 Bounded Tree Width

In proving Theorem 5.2.1, we break the argument into two pieces. In this section, we handle the case in which  $G$  has bounded tree width. Specifically,

**Theorem 5.5.1.** *Let  $G$  be a sufficiently large internally 5-connected graph with bounded tree width and no minor isomorphic to  $K_6$  or  $K_{4,4}$ . Then either  $G$  is apex or*

$G$  contains a peripheral theta.

We make use of many of the ideas and results from [25], specifically the structure of a particular type of tree decomposition of  $G$ , called a path decomposition since the underlying tree is a path.

We state a theorem analogous to Corollary 3.9 in [25] taking (in the context of that Corollary),  $p = 5$  rather than  $p = 6$ . We mention two small differences below.

**Theorem 5.5.2.** *For all integers  $l, w \geq 0$  there exists an integer  $N$  with the following property. If  $G$  is an internally 5-connected graph of tree-width at most  $w$  with at least  $N$  vertices, then either  $G$  has a minor isomorphic to  $K_6$  or  $G$  has a linear decomposition of length at least  $l$  and adhesion at most  $w$  satisfying (L1)-(L9) below.*

We now state the properties (L1)-(L9) that we require and then we mention differences between our properties and those in [25]. We view our linear decomposition of  $G$  as a family of sets  $\mathcal{W} = (W_0, W_1, \dots, W_l)$  such that

(L1)  $\bigcup_{i=0}^l W_i = V(G)$ , and every edge of  $G$  has both ends in some  $W_i$

(L2) If  $0 \leq i \leq j \leq k \leq l$ , then  $W_i \cap W_k \subseteq W_j$

(L3) There is an integer  $q$  such that  $|W_{i-1} \cap W_i| = q$  for all  $i = 1, 2, \dots, l$ . We refer to this property as having *adhesion*  $q$ .

(L4) For every  $i = 1, 2, \dots, l$ ,  $W_{i-1} \neq W_{i-1} \cap W_i \neq W_i$

(L5) There exists a linkage from  $W_0 \cap W_1$  to  $W_{l-1} \cap W_l$  of cardinality  $q$  in which each path is induced

We refer to the linkage given by (L5) as  $\mathcal{P}$ , called the *foundational linkage* with paths called *foundational paths*.

(L6) For all  $i \in \{1, 2, \dots, l-1\}$ , each  $\mathcal{P}$ -bridge of  $G[W_i]$  has attachments in at least 2 members of  $\mathcal{P}$ .

(L7) For every path  $P \in \mathcal{P}$ , if there exists an index  $i \in 1, 2, \dots, l-1$  such that  $P[W_i]$  is a trivial path, then  $P[W_k]$  is a trivial path for all  $k = 1, 2, \dots, l-1$



(L8) For every two distinct paths  $P, P' \in \mathcal{P}$ , if there exists an integer  $k \in \{1, \dots, l-1\}$  such that  $P$  and  $P'$  are bridge adjacent in  $W_k$ , then they are bridge adjacent in  $W_{k'}$  for all  $k' \in \{1, \dots, l-1\}$

(L9) Let  $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{P}$  such that  $|\mathcal{P}_1| + |\mathcal{P}_2| \leq 5$  and each member of  $\mathcal{P}_1$  is non-trivial. Then there exists a linkage  $Q$  in  $G$  of cardinality  $|\mathcal{P}_1|$  from  $W_0 \cap W_1 \cap V(\mathcal{P}_1)$  to  $W_{l-1} \cap W_l \cap V(\mathcal{P}_1)$  such that its graph is a subgraph of  $H := G[W_0 \cup W_l] \cup \bigcup_{P \in \mathcal{P} - \mathcal{P}_2} P$ .

There are only two differences between these properties and those in [25]. First, we choose the linkage in (L6) differently by allowing bridges to neighbor a trivial and a non-trivial path but not to only attach to a non-trivial path. The proof of the argument is nearly identical and follows immediately from Theorem 2.2 in [25]. Second, in (L9), we only have internally 5-connected rather than 5-connected but the proof is exactly identical.

We state the following hypothesis common to several forthcoming lemmas.

**Hypothesis 5.5.3.** *Let  $l \geq 2, q \geq 6, p = 5$  be integers and  $G$  be an internally 5-connected graph with no minor isomorphic to  $K_6$  or  $K_{4,4}$  and no peripheral theta, and let  $\mathcal{W} = (W_0, W_1, \dots, W_l)$  be a linear decomposition of  $G$  of length  $l$  and adhesion  $q$  with a foundational linkage  $\mathcal{P}$  such that conditions (L1)-(L9) hold.*

**Lemma 5.5.4.** *Assume Hypothesis 5.5.3. Then there do not exist  $\binom{4q}{4}$  distinct indices  $i$  with  $1 \leq i \leq l-1$  such that  $G[W_i]$  contains a non-trivial  $\mathcal{P}$ -bridge attaching only to trivial foundational paths.*

*Proof.* Suppose otherwise. Then there exist four distinct indices  $i$  such that  $G[W_i]$  contains a non-trivial  $\mathcal{P}$ -bridge  $B_i$  attaching to the same subset of four trivial foundational paths. By contracting the internal vertices of each  $B_i$  to a single vertex,  $G$  has a  $K_{4,4}$  minor which is a contradiction.  $\square$

**Lemma 5.5.5.** *Assume Hypothesis 5.5.3. If  $l > 4\binom{q}{6}$  then  $\mathcal{P}$  contains at least one non-trivial path*

*Proof.* Suppose otherwise. Then for every  $i, 1 \leq i \leq l-1$ ,  $G[W_i]$  contains a non-trivial bridge  $B_i$  as  $W_i \not\subseteq W_{i+1}, W_i \not\subseteq W_{i-1}$  by (L4), which contradicts Lemma 5.5.4.

□

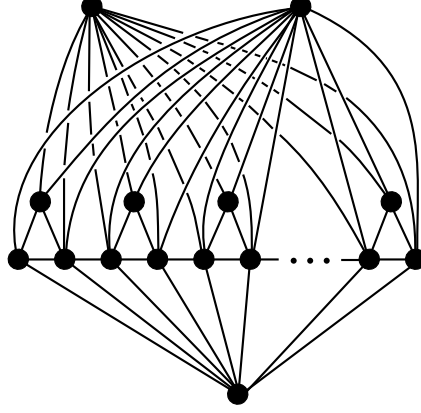


Figure 5.3: Three trivial foundational paths

**Lemma 5.5.6.** *Assume Hypothesis 5.5.3. If  $l \geq 30$ , then no non-trivial foundational path is bridge-adjacent to three or more trivial foundational paths.*

*Proof.* Let  $P$  be the non-trivial foundational path and  $x, y, z$  be the trivial foundational paths. Let  $u \in W_2 \cap W_3 \cap P$  and  $v \in W_{29} \cap W_{30} \cap P$ . Let  $H_{xy}$  be the set of  $P$ -bridges in  $G$  restricted to  $W_3$  through  $W_{30}$  along with  $x, y$ , and  $P$ . By Theorem 5.3.1, either  $H_{xy}$  can be embedded in the plane with  $x, y, u, v$  on the outside or there are vertex disjoint paths between  $x$  and  $y$  and between  $u$  and  $v$ . Suppose the paths exist for each pair from  $\{x, y, z\}$ . Then by property (L9) we can find a path from  $u$  to  $v$  disjoint from all of these things, which immediately gives a  $K_6$  minor.

So we may assume  $H_{xy}$  embeds planarly with  $u, x, v, y$  embedded on the outside in order. Let  $H$  be the largest plane subgraph of  $H_{xy} \cap \bigcup_{i=10}^{20} W_i$  with boundary a cycle and let  $u', v'$  be the first and last vertices of  $P$  in  $H$ . Note that  $P$  divides  $H$  into two planar pieces each with boundary a cycle and  $P$  as part of that boundary. Call these two subgraphs  $H_x$  and  $H_y$  with  $x \in H_x$  and  $y \in H_y$ . Then we claim that not both  $H_x$

and  $H_y$  have vertices not on their outer cycle that neighbor  $z$  in  $G$ . Suppose there were such vertices. Then  $x$  and  $z$  are bridge-adjacent and  $y$  and  $z$  are bridge-adjacent. Look at  $(G - H_{xy} - z) \cup \{u, v, x, y\}$ . Either that graph embeds planarly with  $u, v, x, y$  on the outside, in which case  $G$  is apex or there are vertex-disjoint paths between  $x$  and  $y$  and  $u$  and  $v$  which produces a  $K_6$  minor. So we may assume that  $H_x$  has no interior vertices that neighbor  $z$  in  $G$ .

We know that  $H_x$  is induced since  $H$  and  $P$  are induced. Suppose  $G - H_x$  is not connected. Then there is some component of  $H - H_x$  that attaches only to the outer cycle of  $H_x$ . If it attaches only to  $P$ , then that contradicts (L6). But then it attaches to the outside cycle of  $H$  which is impossible by the maximality of  $H$ . Finally, choose any set  $X$  of size 3 inside  $H_x$  and suppose in  $H_x - X$  there is some component disjoint from the outer cycle of  $H_x$ . Then in  $G$ ,  $X$  is still a cut since no interior vertex of  $H_x$  neighbors  $z$ . So by Lemma 5.4.2,  $G$  contains a peripheral theta which is a contradiction.  $\square$

We make use of the following definition from [25]

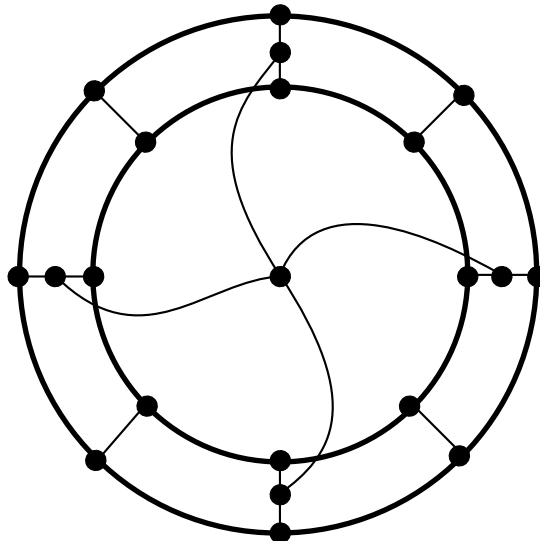


Figure 5.4: A pinwheel with 4 vanes

**Definition.** A *pinwheel* with  $t$  vanes is the graph defined as follows. Let  $C^1$  and  $C^2$  be two disjoint cycles of length  $2t$ , where the vertices of  $C^i$  are  $v_1^i, v_2^i, \dots, v_{2t}^i$  in order. Let  $w_1, w_2, \dots, w_t, x$  be  $t+1$  distinct vertices. The pinwheel with  $t$  vanes has vertex-set  $V(C^1) \cup V(C^2) \cup \{w_1, w_2, \dots, w_t, x\}$  and edge-set  $E(C^1) \cup E(C^2) \cup \{v_{2j}^1 v_{2j}^2 : 1 \leq j \leq t\} \cup \{w_j v_{2j-1}^i : 1 \leq j \leq t, i = 1, 2\} \cup \{xw_j : 1 \leq j \leq t\}$

A *mobius pinwheel* with  $t$  vanes is obtained from a pinwheel with  $t$  vanes by deleting the edges  $v_{2t}^1 v_1^1$  and  $v_{2t}^2 v_1^2$  and adding the edges  $v_{2t}^1 v_1^2$  and  $v_{2t}^2 v_1^1$

Note that a Mobius pinwheel with 4 vanes contains a  $K_6$  minor as shown in [25].

The following lemmas are proven in [25] as Lemma 4.2 and Lemma 5.3 (with 5-connected replaced by internally 5-connected since the proof is identical)

**Lemma 5.5.7.** *Assume Hypothesis 5.5.3. If  $l \geq 30$ ,  $G$  does not contain a non-trivial bridge adjacent to three non-trivial paths.*

**Lemma 5.5.8.** *If an internally 5-connected graph  $G$  with no  $K_6$  minor contains a subdivision of a pinwheel with 20 vanes as a subgraph, then  $G$  is apex.*

This leads us to the following which is analogous to Lemmas 6.4 and 5.1 of [25] and uses the same proof.

**Lemma 5.5.9.** *Assume Hypothesis 5.5.3. If there exist  $\binom{40q}{3}$  distinct indices  $i \in \{1, 2, \dots, l-1\}$  such that  $G[W_i]$  contains a non-trivial  $\mathcal{P}$ -bridge attaching to a trivial foundational path and two non-trivial foundational paths, then  $G$  is apex.*

We define twisting paths as follows. Let  $G$  be a graph and  $\mathcal{W} = (W_0, \dots, W_l)$  be a linear decomposition of length  $l$  and adhesion  $q$  of  $G$ , and let  $\mathcal{P}$  be a foundational linkage such that (L1 – L5) hold. Let  $i \in \{1, 2, \dots, l-1\}$ , let  $P, P' \in \mathcal{P}$  be two non-trivial foundational paths, let  $W_{i-1} \cap W_i \cap V(P) = \{x\}$ ,  $W_{i-1} \cap W_i \cap V(P') = \{x'\}$ ,  $W_i \cap W_{i+1} \cap V(P) = \{y\}$ , and  $W_i \cap W_{i+1} \cap V(P') = \{y'\}$ . Let  $Q_1, Q_2$  be two disjoint paths where  $Q_i$  has ends  $u_i$  and  $v_i$  for  $i = 1, 2$ . If the paths  $Q_1$  and  $Q_2$  are internally

disjoint from  $V(\mathcal{P})$ , the vertices  $x, u_1, u_2, y$  occur on  $P$  in that order, and  $x', v_2, v_1, y'$  occur on  $P'$  in that order, then we say that the foundational paths  $P$  and  $P'$  *twist*.

The following is Lemma 6.3 from [25]

**Lemma 5.5.10.** *Let  $l \geq 2, q \geq 3, p \geq 4$  be integers, and let  $\mathcal{W} = (W_0, W_1, \dots, W_l)$  be a linear decomposition of length  $l$  and adhesion  $q$  of a graph  $G$ , and let  $\mathcal{P}$  be a foundational linkage for  $\mathcal{W}$  such that (L1)-(L5) and (L9) hold. If there exist  $\binom{12q}{2}$  distinct indices  $i \in \{1, 2, \dots, l-1\}$  such that  $G[W_i]$  contains a pair of twisting non-trivial foundational paths, then  $G$  has a  $K_6$  minor.*

This brings us to property

(L10) For every  $i \in \{1, 2, \dots, l-1\}$ , no non-trivial  $\mathcal{P}$ -bridge of  $G[W_i]$  that attaches to two non-trivial paths attaches to any other paths in  $\mathcal{P}$

(L11) No two non-trivial paths twist

**Lemma 5.5.11.** *Assume Hypothesis 5.5.3. If  $l$  is sufficiently large, then there exists a contraction  $\mathcal{W}'$  of  $\mathcal{W}$  of length  $l'$  such that  $\mathcal{W}'$  and the corresponding restriction of  $\mathcal{P}$  satisfy (L1)-(L11)*

*Proof.* By Lemmas 5.5.7, 5.5.9, and 5.5.10 there exists an index  $\alpha$  such that for all  $i \in \{0, 1, \dots, l'\}$  the graph  $G[W_{\alpha+i}]$  does not contain a non-trivial bridge attaching to two non-trivial paths and any other paths of  $\mathcal{P}$  or twisting non-trivial paths. Then clearly the contraction  $(\bigcup_{i=0}^{\alpha-1} W_i, W_\alpha, W_{\alpha+1}, \dots, W_{\alpha+l'}, \bigcup_{i=\alpha+l'+1}^l W_i)$  of  $\mathcal{W}$  is as desired.  $\square$

**Lemma 5.5.12.** *Let  $l \geq 30, q \geq 6$  be integers and  $G$  be an internally 5-connected graph with no  $K_{4,4}$  or  $K_6$  minor, no peripheral theta, and not apex and let  $\mathcal{W} = (W_0, W_1, \dots, W_l)$  be a linear decomposition of  $G$  of length  $l$  and adhesion  $q$  with a foundational linkage  $\mathcal{P}$  such that conditions (L1)-(L11) hold. Then  $G$  does not contain two non-trivial paths that are bridge-adjacent.*

*Proof.* Let  $P$  and  $P'$  be the non-trivial foundational paths. Let  $W_2 \cap W_3 \cap V(P) = \{x\}$ ,  $W_2 \cap W_3 \cap V(P') = \{x'\}$ ,  $W_{29} \cap W_{30} \cap V(P) = \{y\}$ , and  $W_{29} \cap W_{30} \cap V(P') = \{y'\}$ . Let  $H$  be the set of  $\mathcal{P}$  bridges attaching to both  $P$  and  $P'$  in  $G$  restricted to  $W_3$  through  $W_{29}$  along with  $P$  and  $P'$ . By Lemma 5.3.1, either  $H$  can be embedded in the plane with  $x, y, y', x'$  on the outside in order or  $P$  and  $P'$  twist.

So we may assume  $H$  embeds planarly with  $x, y, y', x'$  embedded on the outside in order. Let  $H'$  be the largest plane subgraph of  $H \cap \bigcup_{i=10}^{20} W_i$  with boundary  $PQ_1P'Q_2$  where  $Q_1$  and  $Q_2$  are paths in  $H$  disjoint from  $P$  and  $P'$  except at their ends.

We know that  $H'$  is induced since  $H$ ,  $P$  and  $P'$  are induced. Suppose  $G - H'$  is not connected. Then there is some component of  $H - H'$  that attaches only to the outer cycle of  $H'$ . If it attaches only to  $P$  or  $P'$ , then that contradicts (L6). But then it attaches to the outside cycle of  $H'$  which is impossible by the maximality of  $H'$ . Finally, choose any set  $X$  of size 3 inside  $H'$  and suppose in  $H' - X$  there is some component disjoint from the outer cycle of  $H'$ . Then in  $G$ ,  $X$  is a cut since by property (L10) no interior vertex of  $H'$  has neighbors outside of  $H'$ . So by Lemma 5.4.2,  $G$  contains a peripheral theta which is a contradiction.  $\square$

We are now prepared to prove Theorem 5.5.1

**Theorem 5.5.13.** *For every  $k \geq 0$  there exists an integer  $N$  such that if  $G$  is an internally 5-connected graph on at least  $N$  vertices with tree width at most  $k$  and no minor isomorphic to  $K_6$  or  $K_{4,4}$  then either  $G$  is apex or  $G$  contains a peripheral theta.*

*Proof.* By Lemmas 5.5.2 and 5.5.11,  $G$  has a linear decomposition satisfying properties (L1)-(L11) with linkage  $\mathcal{P}$ . By Lemma 5.5.5,  $\mathcal{P}$  contains a non-trivial path. Since  $G$  is internally 5-connected,  $\mathcal{P}$  must be bridge-adjacent to either 3 trivial paths or another trivial path, both of which are impossible by Lemmas 5.5.6 and 5.5.12 which is a contradiction.  $\square$

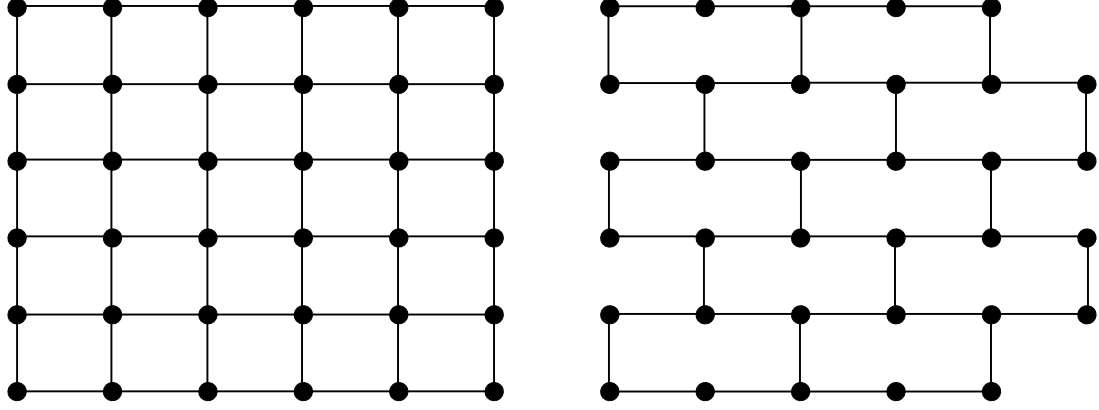


Figure 5.5: A grid and a wall

## 5.6 Large Tree Width

Suppose instead that  $G$  has large tree width. Then we will look for some structure inside  $G$  that either contains a peripheral theta, a  $K_6$  minor, or a large induced planar subgraph. We define first the notion of a *wall*, using the definition from [27]. Let  $[r]$  denote  $\{1, 2, \dots, r\}$  and let  $r \geq 2$  be an integer. An  $r \times r$ -grid is the graph with vertex-set  $[r] \times [r]$  in which  $(i, j)$  is adjacent to  $(i', j')$  if and only if  $|i - i'| + |j - j'| = 1$ . An *elementary  $r$ -wall* is obtained from the  $2r \times r$ -grid by deleting all edges with ends  $(2i - 1, 2j - 1)$  and  $(2i - 1, 2j)$  for all  $i = 1, 2, \dots, r$  and  $j = 1, 2, \dots, \lfloor r/2 \rfloor$  and all edges with ends  $(2i, 2j)$  and  $(2i, 2j + 1)$  for all  $i = 1, 2, \dots, r$  and  $j = 1, 2, \dots, \lfloor (r - 1)/2 \rfloor$  and then deleting the two resulting vertices of degree one. An  $r$ -wall is any graph obtained from an elementary  $r$ -wall by subdividing edges.

This leads us to Theorem 1.6 of [27]:

**Theorem 5.6.1.** *Let  $t \geq 5$  and  $r \geq 3\lceil\sqrt{t}\rceil$  be integers. Let  $n = 12288t^{24}$ ,  $R = r^{2n}$ , and  $R_0 = 49152t^{24}(40t^2 + R)$ . Let  $G$  be a graph, and let  $W_0$  be an  $R_0$ -wall in  $G$ . Then either  $G$  contains a  $K_t$  minor or there exists a set  $A \subseteq V(G)$  of size at most  $t - 5$  and an  $r$ -subwall  $W$  of  $W_0$  such that  $V(W) \cap A = \emptyset$  and  $W$  is a flat wall in  $G - A$ .*

In an internally 4-connected graph,  $G$ , a flat wall is a wall  $W$  with outer cycle  $C$  such that there is a separation  $(A, B)$  of  $G$  such that  $A \cap B \subseteq V(C)$ ,  $V(W) \subseteq B$  and

$G[B]$  can be drawn on a disk with  $C$  as its outer cycle. We say that the *surface* of  $W$  is  $G[B]$ .

The following version of 5.6.1 is more convenient for our purposes:

**Theorem 5.6.2.** *For every integer  $r > 1$ , there exists integers  $k$  and  $n$  such that the following is true. Let  $G$  be an internally 5-connected graph of tree width at least  $k$  on at least  $n$  vertices. Then either  $G$  contains a  $K_6$  minor or there exists a vertex  $v \in V(G)$  such that  $G - \{v\}$  contains a flat  $r$ -wall.*

Since we prefer to work with planar rather than apex graphs, we would like to refine the ways in which  $v$  attaches to  $W$ .

**Lemma 5.6.3.** *Let  $G$  be an internally 5-connected graph,  $v \in V(G)$ ,  $r \geq 1000$  be an integer, and let  $W$  be a flat  $r$ -wall in  $G - \{v\}$ . Let  $r' = \lfloor r/100 \rfloor$ . Then either there exists a flat  $r'$ -wall of  $G - \{v\}$ ,  $W'$  with no vertex in the surface of  $W'$  adjacent to  $v$  in  $G$  or  $G$  contains a subdivision of a pinwheel with 20 vanes as a subgraph.*

*Proof.* Partition the vertices of  $W$  in the natural way into 100 subwalls in a  $10 \times 10$  grid,  $W_{ij}, 1 \leq i, j \leq 10$ . Since each of the  $W_{ij}$  is a flat wall of  $G - \{v\}$ , we may assume that some vertex in the surface of  $W_{ij}$  neighbors  $v$  in  $G$ . Take  $C_1$  to be the outer cycle of  $W$  and  $C_2$  to be the outer cycle of the subwall of  $W$  formed by the union of  $W_{ij}, 3 \leq i, j \leq 8$ . Let  $u_{ij}$  be the vertex in the surface of  $W_{ij}$  that is adjacent to  $v$ . Then by Menger's Theorem,  $u_{ij}$  has vertex disjoint paths to  $C_1$  and  $C_2$  that stay entirely inside  $W_{ij}$  for  $i = 2, 9, 2 \leq j \leq 9$  or  $j = 2, 9, 2 \leq i \leq 9$ . By including  $v$  and every other edge between  $v$  and a  $u_{ij}$ , this is a subdivision of a pinwheel with 20 vanes.  $\square$

This almost immediately gives us the main theorem of this section

**Theorem 5.6.4.** *There exist integers  $k$  and  $n$  such that if  $G$  is an internally 5-connected graph with tree width at least  $k$  and at least  $n$  vertices with no minor isomorphic to  $K_6$  or  $K_{4,4}$  then  $G$  contains a peripheral theta.*



*Proof.* Choose  $k$  and  $n$  as in Theorem 5.6.2 for  $r = 1000$ . Then  $G$  contains a vertex  $v$  and a flat  $r$ -wall in  $G - \{v\}$ . By Lemma 5.6.3 and Lemma 5.5.8,  $G$  contains a flat  $r'$ -wall,  $W$ ,  $r' = \lfloor r/100 \rfloor$ , with no vertex in the surface of  $W$  adjacent to  $v$ . Let  $W'$  be an  $r' - 2$  subwall of  $W$  that uses no vertices of the outer cycle of  $W$ . Let  $C$  be the outer cycle of  $W$  and  $H$  be the component of  $G - \{v\} - C$  that contains  $W'$ . By our choice of  $H$ ,  $H$  is induced and  $G - H$  is connected. Since no vertex of  $H$  neighbors  $v$ , there is no set of size three,  $X$ , inside  $H$  such that  $H - X$  has a component disjoint from the outer cycle. By Lemma 5.4.2,  $G$  contains a peripheral theta.  $\square$

## 5.7 The Main Theorem (revisited)

We now combine the previous two results and then restate and prove the main theorem of this chapter

**Theorem 5.7.1.** *There exists an absolute constant  $N$  such that every internally 5-connected graph  $G$  on at least  $N$  vertices with no minor isomorphic to  $K_6$  or  $K_{4,4}$  either is apex or contains a peripheral theta graph.*

*Proof.* Let  $k$  and  $n_1$  be the values from Theorem 5.6.4 such that if  $G$  is an internally 5-connected graph with tree width at least  $k$  and at least  $n_1$  vertices with no minor isomorphic to  $K_6$  or  $K_{4,4}$  then  $G$  contains a peripheral theta. Let  $n_2$  be the values from Theorem 5.5.1 such that if  $G$  is an internally 5-connected graph on at least  $n_2$  vertices with tree width at most  $k$  and no minor isomorphic to  $K_6$  or  $K_{4,4}$  then either  $G$  is apex or  $G$  contains a peripheral theta. Let  $N$  be the larger of  $n_1$  and  $n_2$ . Then since the tree-width of  $G$  is either at most  $k$  or at least  $k$ , we are done.  $\square$

Which leads to the main theorem of the chapter.

**Theorem 5.7.2.** *There exists an absolute constant  $N$  such that if  $G$  is a graph on at least  $N$  vertices at least one of the following holds:*

- (1)  $G$  contains a graph in the Petersen family as a minor

- (2)  $G$  contains a complete separation
- (3)  $G$  contains a peripheral theta, the deletion of whose arc leaves  $G$  Kuratowski-connected
- (4) There exists  $X \subseteq V(G)$ ,  $|X| \leq 1$  such that  $G \setminus X$  is planar

*Proof.* If  $G$  is not internally 5-connected, then it either contains a small separation ( $\leq 3$ ) or contains a separation of size 4. All separations in the former case are complete. In the latter case, we apply Theorem 5.4.7 to find either outcome 1, 2, or 3 above. So we may assume  $G$  is internally 5-connected and not apex. But then applying Theorem 5.7.1 gives outcome 1 or outcome 3.  $\square$

## 5.8 An Alternative Approach

In the above sections we discuss a possible approach to finding a flat embedding for a graph that makes use of peripheral thetas. Another possibility is to find a planar subgraph, contract an edge, and recur. Then, when we have a flat embedding for the contracted graph, find the sphere on which the planar subgraph is embedded and uncontract the edge in a planar way on that sphere. Ignoring the difficulties in finding the sphere, this seems, on the face of it reasonable. We formalize the notions as follows:

**Definition.** Let  $G$  be a graph. Let  $H$  be a subgraph of  $G$ . If there exist a set of vertices  $X \subseteq V(H)$  such that  $E(H)$  contains all neighbors of  $X$  and there is a planar embedding of  $H$  whose outer cycle consists of exactly the vertices of  $V(H) \setminus X$ , we say that  $H$  is locally planar about  $X$  and this embedding is the *representative embedding*. If  $H$  is connected,  $|X| \geq 2$  and  $X$  contains at least two vertices that are adjacent in  $G$ , we say that  $H$  is an *eye* of  $G$  with *pupil*  $X$  and *iris*  $V(H) \setminus X$ .

A nice companion to Theorem 5.2.1 would then be that every sufficiently large, internally 5-connected flat graph contains an eye. This statement, however, seems

more difficult. There are several dangers in the case of small cutsets, but there is a more significant danger in trying to prove an analogue of Lemma 5.5.6. Consider the following graph. Take an even path  $P = v_1, v_2, \dots, v_n$ . Add three vertices  $a, b, c$  adjacent to all the vertices of  $P$  and an edge between  $a$  and  $b$ . We then add two vertices  $u$  and  $w$  with  $u$  adjacent to  $v_1, v_n, b, c$  and  $w$  adjacent to  $v_1, v_2, a, c$ . Finally, for  $i$  odd,  $1 \leq i \leq n$  we add a vertex  $x_i$  adjacent to  $v_i, v_{i+1}, a, b$ . We will refer to a graph built from this construction as a tribranch and show an example in Figure 5.6.

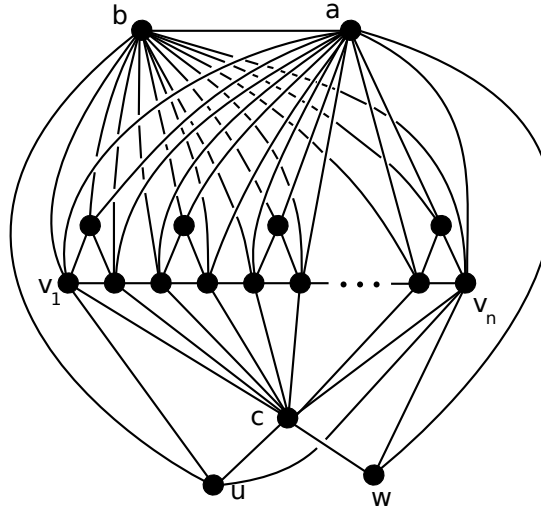


Figure 5.6: An embedded tribranch

We note first that tribranches are internally 5-connected. Any cut set of size at most 4 other than the neighbors of a degree four vertex would need to include  $a, b$  and  $c$ . But then the resulting graph is at least two connected since every vertex is contained in a cycle. We show next that tribranches are flat.

**Lemma 5.8.1.** *Tribranches are flat.*

*Proof.* We show instead that tribranches are linkless and appeal to Theorem 5.1.1. We consider the embedding depicted in Figure 5.6. Let  $G$  be a tribranch. Then we delete vertex  $b$  and the edge between  $u$  and  $v_n$  and embed the resulting graph planarly. Add back vertex  $b$  as an apex and then add the  $u - v_n$  edge as depicted (with the

$u - v_n$  edge on the side of the  $c - w$  edge towards  $b$ ). Note that the embedding without one of the  $u - v_n$  edge or the  $c - w$  edge is linkless since it is the canonical apex embedding. Further, contracting the  $u - b$  edge or the  $a - w$  edge would similarly result in a linkless embedding for the same reason. Finally,  $a$  and  $b$  are symmetrical in regards to this embedding.

We then consider two vertex disjoint cycles. Both the  $u - v_n$  edge and the  $c - w$  edge must be used, since otherwise their linking number is 0 since they are two cycles in the apex embedding. Similarly, if one cycle uses both edges, we can imagine a degree four vertex at their intersection and break that cycle into two pieces. The linking number of each piece with the other cycle must be 0 since the resulting embedding is the canonical apex one, so the linking number of the original two cycles must be 0 as well. So one of the cycles must use the  $u - v_n$  edge and the other must use the  $c - w$  edge.

Note that if either cycle uses the  $u - b$  edge or the  $w - a$  edge, their linking numbers must be 0, since then contracting the  $u - b$  edge (or  $w - a$  edge) gives two cycles in the canonical apex embedding. It is clear that uncontracting does not change their linking number here, so it must be 0.

Let  $C$  be the cycle that uses the  $u - v_n$  edge and  $C'$  the cycle that includes the  $w - c$  edge. Then  $C$  contains another edge incident with  $u$  and not the  $u - b$  edge, so contains  $u - v_1$ . But then  $C'$  contains an edge incident with  $w$  which cannot be  $w - a$  and does not contain  $v_1$  or  $v_n$ , which is impossible, so this embedding must be linkless.  $\square$

So tribranches are flat, 5-connected graphs. We are now interested in showing that tribranches do not contain eyes whose pupil does not contain  $u$  or  $w$ . In an analogous proof to that of Lemma 5.5.6, we only have control over the vertices other than  $u$  and  $w$ , so this suffices as a counterexample to that particular technique. To do so, we show that, for each of these edges, there is no locally planar subgraph about

its ends. We require the following lemma

**Lemma 5.8.2.** *Let  $G$  be a 4-connected graph with an edge  $uv$  in 3 triangles. Then  $G$  does not contain an eye with  $u$  and  $v$  in its pupil.*

*Proof.* Suppose otherwise. Then there is an eye  $H$  with pupil  $X$  containing  $u$  and  $v$ . Let  $a, b, c$  be the neighbors of both  $u$  and  $v$ . Then  $\{a, b, c\} \subset V(H)$ . Look at a representative embedding of  $H$  with  $u$  and  $v$  not in the outer cycle. Since  $K_{3,3}$  is not planar, in such an embedding at least one of the triangles  $auv, buv, cuv$  must be separating. Without loss of generality, we say it is  $auv$ . Let  $S$  be the set of vertices in the side of this separation disjoint from the outer cycle. Then all of the neighbors of vertices of  $S$  in  $G$  must be in  $H$  by the definition of an eye, but then  $\{a, u, v\}$  is a separation in  $G$  which is a contradiction.  $\square$

We now consider the edges of a tribranch labelled as above and note that, except for edges whose ends are  $u$  or  $w$ , the only edges not in three triangles are from  $c$  to  $v_i$ , for  $2 \leq i \leq n-1$ . But consider such an edge and note that the pupil of an appropriate eye can only contain  $c$  and  $v_i$ . Then  $v_{i-1}$  and  $v_{i+1}$  form a cutset divide the outer cycle of the eye into two pieces; those adjacent to  $c$  and those adjacent to  $v_i$ . So on one side lie  $a$  and  $b$  and on the other lie all of the other  $v_j$  as well as  $u$  and  $w$ . But  $u$  and  $w$  only have two neighbors between them other than  $a, b, c$ , so there is no path through all of the  $v_j$  as well as  $u$  and  $w$  that avoids  $a, b$ , and  $c$ , so there can be no such eye.

Tribranches, then, do not contain eyes whose pupil does not contain  $u$  or  $w$ . If we wanted to prove an analogue for Theorem 5.2.1, the proof technique used throughout this chapter would not work. The vertices  $u$  and  $w$  are not within the bags considered as part of the linkage, so we have very little control over them; it seems unlikely that we would be able to guarantee that the eyes which contain them are actually eyes in the underlying graph. This serves to cast some doubt on the statement above, that sufficiently large, internally 5-connected flat graphs contain eyes, but, even more,

makes it very unlikely that any proof analogous to the one used for Theorem 5.2.1 could work.

# CHAPTER VI

## SERPENTS

### 6.1 Introduction

While the previous chapter was concerned with general, large flat graphs, this chapter is concerned with one particular structure. As noted previously, several classes of graphs have natural flat embeddings. For instance, planar graphs can be embedded planarly. Apex graphs can be embedded with the planar piece in a plane and the apex above it with straight lines to its neighbors. Both of these graphs are nearly planar in the sense that after deleting a small number of vertices (one in each case), the resulting graph is planar. A reasonable question might be to ask whether there is some fixed  $k$  such that all reasonably well-connected flat graphs can be made planar by deleting at most  $k$  vertices. This, however, is untrue. We present in this chapter a class of flat graphs that are 5-connected such that, for any  $k$ , there are infinitely many graphs in our class which are not planar after deleting any  $k$  vertices. Specifically,

**Definition.** Let  $G$  be a graph defined by the following construction. For  $i \in \{1, \dots, n\}$ , let  $H_i$  be a plane graph with outer cycle  $C_i$ . For each  $H_i$ , let  $v_{i1}, v_{i2}, \dots, v_{i10}$  be vertices in order around  $C_i$ . For each  $i \in \{1, \dots, n-1\}$ , add matchings between  $\{v_{i6}, v_{i8}, v_{i10}\}$  and  $\{v_{(i+1)1}, v_{(i+1)3}, v_{(i+1)5}\}$  and between  $\{v_{i7}, v_{i9}\}$  and  $\{v_{(i+1)2}, v_{(i+1)4}\}$ . We call such a graph a *snakelet*, the union of the matchings a *transition*, and the  $H_i$  the *scales*.

We extend this definition, by allowing a special piece on either end:

**Definition.** Let  $G$  be a snakelet. We augment  $G$  as follows. Let  $P$  be a planar graph and  $H$  be either the first or last scale of  $G$ . Let  $v_1, \dots, v_5$  be vertices on the outer cycle

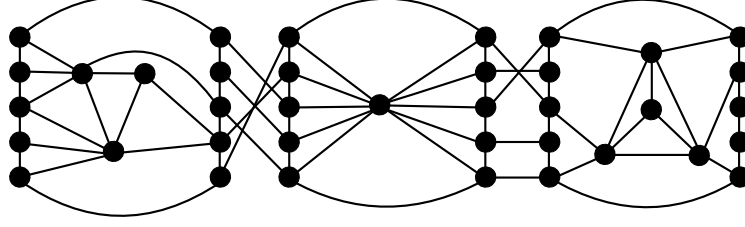


Figure 6.1: A snakelet

of  $H$  in order, such that there is a path on the outer cycle of  $H$  between  $v_1$  and  $v_5$  disjoint from a transition of  $H$ . We say that  $P$  is a *tail* of Type 1 if there is a vertex of  $P$ ,  $u_1$  and 4 vertices in order on the outer cycle of  $P$ ,  $u_2, u_3, u_4, u_5$  such that there is a matching between  $\{u_1, u_3, u_5\}$  and  $\{v_1, v_3, v_5\}$  and a matching between  $\{u_2, u_4\}$  and  $\{v_2, v_4\}$ . We say that  $P$  is a *tail* of Type 2 if there exist nine vertices on the outer cycle of  $P$  in order,  $u_1, \dots, u_9$  such that there is a matching between  $\{u_1, u_3, u_5\}$  and  $\{v_1, v_3, v_5\}$  and a matching between  $\{u_2, u_4\}$  and  $\{v_2, v_4\}$ . and  $v_7v_9$  is an edge and  $v_6v_8$  is an edge. We say that a graph  $G'$  is a *serpent* if  $G'$  contains a snakelet with at most two *tails*, at most one attached to the first scale and at most one attached to the last scale.

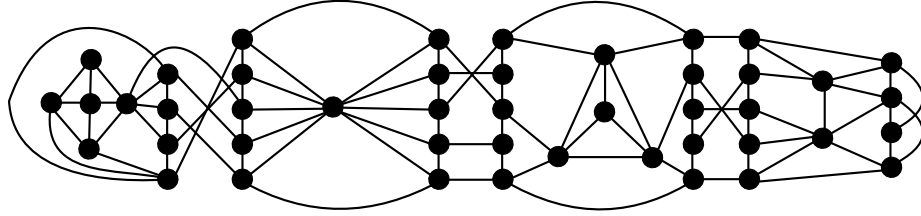


Figure 6.2: A serpent

We now intend to prove the following theorem:

**Theorem 6.1.1.** *Serpents are flat.*

We note first that, by choosing the interior planar graphs to be 5-connected, we can make our serpents 5-connected as well. In order to prove Theorem 6.1.1, we



instead prove that serpents are linkless, which proves that they are flat by Theorem 5.1.1. We will perform several computations in this chapter on the linking number, defined in Section 1.4.

## 6.2 Serpents

We will prove Theorem 6.1.1 by providing an embedding for every serpent. We begin by handling snakelets. Specifically, take a snakelet, embed each  $H_i$  planarly as specified in the definition in order from left-to-right by index. If all of the transitions were simply planar, this would then be a planar embedding. Then it remains to provide embeddings for each of the transitions. We may assume that none of the transitions are planar, since otherwise we can view the two scales joined by a planar transition as one larger scale. Note that we may assume that we have edges  $v_{i,9}v_{(i+1),2}$  and  $v_{i,7}v_{(i+1),4}$  in each transition by our choice of ordering  $C_i$  in the definition of  $G$ . There are, therefore, 5 different possible transitions, listed in Figure 6.3. The embeddings of each of the transitions given in the figures we will refer to as  $E_1, \dots, E_5$ . We will refer to the embedding for each transition defined by, for each intersection, choosing the opposite edge to go over the other as  $\overline{E}_1, \dots, \overline{E}_5$ . For a vertex  $v$  in a transition  $T$  we will refer to  $T(v)$  as the neighbor of  $v$  in the transition.

For the ease of notation, for  $H_i$  a scale, we will refer to  $u_{i,1} = v_{i,10}, u_{i,2} = v_{i,9}, \dots, u_{i,5} = v_{i,6}$  as the right transfer,  $R$ , as well as to  $v_{i,1}, \dots, v_{i,5}$  as the left transfer,  $L$ . A transition is then a matching (of the particular type defined above) between the right transfer of scale  $H_i$  to the left transfer of scale  $H_{i+1}$ .

**Definition.** Let  $P$  be a path in  $G$  with both ends in  $H_i$  for some  $i$  and let  $j_0 \leq i$  be an integer such that  $|E(P) \cap E(T_j)| = 2$  for each transition  $T_j$ ,  $j_0 \leq j < i$  and such that  $P$  is contained in  $\bigcup_{j=j_0}^i H_i$ . Let  $(c_{j_0}, d_{j_0}), (a_{j_0+1}, b_{j_0+1}), (c_{j_0+1}, d_{j_0+1}), \dots, (a_i, b_i)$  be a sequence of pairs of integers such that the two edges of  $E(P) \cap T_j$  are  $u_{c_j}v_{a_j}$

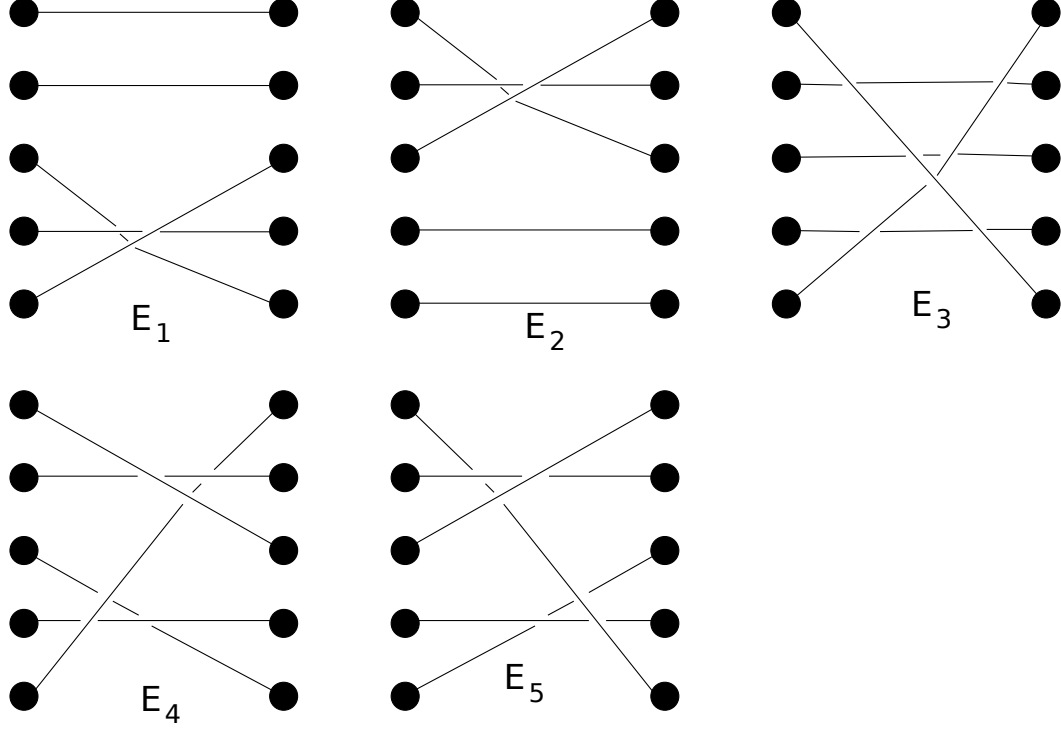


Figure 6.3: The five transitions with embeddings

and  $u_{d_j}v_{b_j}$ . Then we say that  $P$  is a *pseudocycle* with twists  $(c_{j_0}, d_{j_0}), (a_{j_0+1}, b_{j_0+1}), (c_{j_0+1}, d_{j_0+1}), \dots, (a_i, b_i)$ . We refer to the twists  $(c_j, d_j)$  as *right twists* and to the twists  $(a_j, b_j)$  as *left twists*. The *vertices* of a left twist  $(a_j, b_j)$  are  $v_{ja_j}$  and  $v_{jb_j}$  and of a right twist  $(c_j, d_j)$  are  $u_{jc_j}$  and  $u_{jd_j}$ .

We are interested in pairs of vertex disjoint cycles and would like to instead discuss pairs of vertex disjoint pseudocycles. Since pseudocycles may only use two edges of each transition, we need to take care of cycles that use four edges of each transition. We will refer to those cycles that use at most 2 edges of each transition as *lean*.

**Lemma 6.2.1.** *Let  $C_1$  and  $C_2$  be vertex disjoint cycles in a serpent  $G$  with scales  $H_1, \dots, H_i$ . Then there exist a pair of vertex disjoint lean cycles,  $C'_1$  and  $C'_2$  so that  $C'_1$  and  $C'_2$  have the same linking number as  $C_1$  and  $C_2$ .*

*Proof.* Choose  $C_1$  and  $C_2$  to be the vertex disjoint cycles with the desired linking number such that the number of transitions in which one of  $C_1$  or  $C_2$  uses four edges

is minimum. We may assume  $G$  has at least two scales since otherwise  $C_1$  and  $C_2$  are both lean. If there is no transition such that  $C_1$  and  $C_2$  both use edges of that transition, we are done since their linking number is 0 and  $G$  contains vertex disjoint lean cycles with linking number zero (for example the outer cycles of two different scales). Let  $i_0 \leq i_1$  be integers such that  $H_{i_0}$  and  $H_{i_1}$  both contain vertices of both  $C_1$  and  $C_2$  and so that  $i_1 - i_0$  is as large as possible. Note that for  $i_0 \leq i \leq i_1$ ,  $H_i$  contains vertices of both  $C_1$  and  $C_2$  since  $C_1$  and  $C_2$  are connected. By construction,  $C_1$  and  $C_2$  use exactly two edges each of  $T_{i_0}, \dots, T_{i_1-1}$ . We may assume, by symmetry that  $C_1$  is not lean, that it uses four edges of a transition  $T_j$  with  $j \leq i_0$ , and that both  $C_1$  and  $C_2$  use two edges of  $T_{i_0}$ .

Let  $x_1$  and  $x_2$  be the vertices of  $H_{i_0}$  incident with the two edges of  $T_{i_0} \cap C_1$ . Then let  $P_1, P_2$  be the shortest subpaths of  $C_1$  contained in  $H_{i_0}$  with end, respectively,  $x_1$  or  $x_2$  and the other end, respectively,  $y_1$  or  $y_2$ ,  $y_1, y_2 \in V(T_{i_0-1}) \cap V(H_{i_0})$ . Let  $z_1, z_2$  be respectively the neighbors of  $y_1, y_2$  in  $H_{i_0-1}$ . Let  $P_3$  be the subpath of  $C_1$  between  $x_1$  and  $x_2$  that contains two edges of  $T_{i_0}$  and let  $P_4$  be a subpath of the outer cycle of  $H_{i_0-1}$  between  $z_1$  and  $z_2$ . Let  $C'_1 = x_1 P_3 x_2 P_2 z_2 P_4 z_1 P_1 x_1$ . Then  $C'_1$  and  $C_2$  have the same linking number as  $C_1$  and  $C_2$  since  $C'_1$  is identical to  $C_1$  on any transitions used by both  $C_1$  and  $C_2$ , but  $C'_1$  has fewer transitions in which it uses four edges of that transition which contradicts our choice of  $C_1$  and  $C_2$ .  $\square$

We will show that every pair of vertex disjoint lean cycles has linking number zero, so every pair of vertex disjoint cycles will as well.

**Definition.** Let  $P_1$  and  $P_2$  be two pseudocycles that are vertex-disjoint. Then we refer to  $(P_1, P_2)$  as a *pseudopair*. A *twist* in a pseudopair is a pair of tuples  $(a, b), (c, d)$  for which there exists an  $i$  such that  $(a, b)$  is the right twist of  $P_1$  in  $H_i$  and  $(c, d)$  is the right twist of  $P_2$  in  $H_i$ . The *vertices* of a twist are the vertices of the corresponding right twists in  $P_1$  and  $P_2$ . We will refer to the *final twist* of  $(P_1, P_2)$  to be the twist of  $(P_1, P_2)$  in the transition of highest index containing edges of both  $P_1$  and  $P_2$ .

**Definition.** Let  $T$  be a transition and  $a, b \in [1, 5]$  be distinct. Let  $u_a, u_b$  be the corresponding vertices of the right transfer of  $T$  and let  $e_a, e_b$  be the edges  $u_a T(u_a), u_b T(u_b)$ . Then  $\phi_1(e_a, e_b) = 0$  if  $e_a$  and  $e_b$  do not cross,  $+1$  if they do and  $e_a$  crosses under  $e_b$ , and  $-1$  if they cross and  $e_a$  crosses over  $e_b$ . Let  $\phi_2(e_a, e_b)$  be  $1$  if  $b > a$  and  $-1$  if  $b < a$ . Let  $\phi_T(a, b) = \phi_1(e_a, e_b)\phi_2(e_a, e_b)$ .

We refer to  $\phi_T$  as the *transition function* for  $T$  and when the transition is understood, we refer to it simply as  $\phi$ .

**Definition.** Let  $(P_1, P_2)$  be a pseudopair with a twist  $((a, b), (c, d))$  with vertices  $v_a, v_b, v_c, v_d$ . Let  $e_a$  be the edge between  $v_a$  and  $T(v_a)$  and similarly for  $e_b, e_c, e_d$ . Then we define the function  $\text{lnk}((a, b), (c, d)) = \phi(e_a, e_c) + \phi(e_b, e_d) - \phi(e_b, e_c) - \phi(e_a, e_d)$ , which we call the link of the twist.

Similarly, we refer to the link of a pseudopair as the sum of the links of each of its twists.

Let  $C_1$  and  $C_2$  be cycles formed by adding an edge between the two ends of  $P_1$  and  $P_2$  respectively. Then the linking number of  $C_1$  and  $C_2$  is exactly one half the link of the pseudopair  $(P_1, P_2)$  plus the contribution from crossings containing the newly added edges. This follows immediately from the definition of the link and the fact that the only crossings between edges of  $C_1$  and  $C_2$  occur at twists (again, ignoring the crossings from the two newly added edges).

**Definition.** Let  $(P_1, P_2)$  be a pseudopair with twist  $((a, b), (c, d))$ . Then we say that the twist is *planar* if there exists a planar graph on 5 vertices with outer cycle  $v_1, v_2, \dots, v_5$  and edges  $v_a v_b$  and  $v_c v_d$ .

**Definition.** Let  $(P_1, P_2)$  be a pseudopair with twist  $((a, b), (c, d))$ . Then the *parity* of the twist is  $0$  if it is planar. If it is non-planar, the parity is  $1$  if  $a - b$  and  $c - d$  have the same sign and  $-1$  otherwise.

Note that the parity of twist  $i$  within a pseudopair is entirely determined by twist  $i - 1$  and the corresponding transition  $T_i$ . Similarly, the link of any twist  $((a, b), (c, d))$  in a pseudopair is determined by the embedding of the transition on the edges with one end at  $v_a, v_b, v_c, v_d$ .

Instead of discussing lean cycles, we intend to discuss pseudopairs. The following lemma will allow this:

**Lemma 6.2.2.** *Let  $C_1$  and  $C_2$  be vertex-disjoint lean cycles in a snakelet  $G$ . Then there exists a pseudopair  $(P_1, P_2)$  with planar final twist whose link is twice the linking number of  $C_1$  and  $C_2$ .*

*Proof.* Choose  $C_1$  and  $C_2$  so that the number of scales that contain a vertex of one but not the other is minimum. We may assume that there is some scale containing vertices of both cycles since otherwise we are done. If there is such a scale  $H_i$ , say it contains vertices of  $C_1$  but not  $C_2$ . Suppose  $C_2$  has vertices in some scale  $H_j$ ,  $j > i$ . Then  $C_1$  uses exactly two vertices of  $H_i \cap T_i$ , say  $v_a, v_b$ . Then we can replace the subpath of  $C_2$  between  $v_a$  and  $v_b$  that does not contain vertices of  $H_{i+1}$  with a subpath of the outer cycle of  $H_i$  between  $v_a$  and  $v_b$ . So we may assume that there are at most two scales containing vertices of one of  $C_1$  and  $C_2$  and not the other and that in those scales, the subpath of the corresponding cycle is just a path in the outer cycle of the scale.

Delete an edge of  $C_1$  in the scale of highest index,  $H_i$  that contains two of its vertices and do the same for  $C_2$  to find paths  $P_1$  and  $P_2$ . Then  $(P_1, P_2)$  is a pseudopair. Since the edges deleted from  $C_1$  and  $C_2$  did not cross (since they were edges of a scale) and since the linking number of  $C_1$  and  $C_2$  is determined solely by crossings on transitions, then by the definition of the linking number and the link, the link of  $(P_1, P_2)$  is half the linking number of  $C_1$  and  $C_2$ . Further,  $(P_1, P_2)$  must have planar final twist since contracting the edges of  $C_1$  and  $C_2$  contained in  $H_i$  (and possibly  $H_{i+1}$  or  $H_{i-1}$ ) gives the desired planar graph.  $\square$

Right	Left	Parity	Link
$(1, 2), (3, 4)$	$(1, 2), (5, 4)$	0	0
$(1, 2), (3, 5)$	$(1, 2), (5, 3)$	0	0
$(1, 2), (4, 5)$	$(1, 2), (4, 3)$	0	0
$(1, 3), (4, 5)$	$(1, 5), (4, 3)$	0	0
$(1, 4), (2, 3)$	$(1, 4), (2, 5)$	+	−
$(1, 5), (2, 3)$	$(1, 3), (2, 5)$	+	−
$(1, 5), (2, 4)$	$(1, 3), (2, 4)$	+	−
$(1, 5), (3, 4)$	$(1, 3), (5, 4)$	0	0
$(2, 3), (4, 5)$	$(2, 5), (4, 3)$	0	0
$(2, 5), (3, 4)$	$(2, 3), (5, 4)$	0	0
$(1, 3), (2, 4)$	$(1, 5), (2, 4)$	−	−
$(1, 3), (2, 5)$	$(1, 5), (2, 3)$	−	−
$(1, 4), (2, 5)$	$(1, 4), (2, 3)$	−	−
$(1, 4), (3, 5)$	$(1, 4), (5, 3)$	−2	0
$(2, 4), (3, 5)$	$(2, 4), (5, 3)$	−2	0

Figure 6.4: Parity and links for  $E_1$

The next lemma follows immediately from the definition of  $\text{lnk}$ .

**Lemma 6.2.3.** *Let  $X = ((a, b), (c, d))$  be a twist. Then  $\text{lnk}(X) = \text{lnk}((c, d), (a, b)) = -\text{lnk}((d, c), (a, b))$*

We present information about each transition in Figures 6.4 through 6.8. The first column lists, up to permutation, all of the possible right twists for a pseudopair. The top portion of each table corresponds to those pseudopairs that have zero parity and the bottom portion corresponds to those that have parity +1. The second column shows the resulting left twist. The third shows the change in parity between those two twists, and the fourth shows the link of the right twist in that transition.

**Lemma 6.2.4.** *There is a choice of embeddings for each of the transitions of  $G$ ,  $T_1, \dots, T_n$ , chosen from either  $E_i$  or  $\overline{E}_i$  for the appropriate  $i \in \{1, 2, 3, 4, 5\}$  as shown above with the following two properties. Any pseudopair with a planar final twist has link 0. The link of every pseudopair with non-planar final twist is either 1 or −1 and*

Right	Left	Parity	Link
(1, 2), (3, 4)	(3, 2), (1, 4)	0	0
(1, 2), (3, 5)	(3, 2), (1, 5)	0	0
(1, 2), (4, 5)	(3, 2), (4, 5)	0	0
(1, 3), (4, 5)	(3, 1), (4, 5)	0	0
(1, 4), (2, 3)	(3, 4), (2, 1)	0	0
(1, 5), (2, 3)	(3, 5), (2, 1)	0	0
(1, 5), (2, 4)	(3, 5), (2, 4)	+	—
(1, 5), (3, 4)	(3, 5), (1, 4)	+	—
(2, 3), (4, 5)	(2, 1), (4, 5)	0	0
(2, 5), (3, 4)	(2, 5), (1, 4)	+	—
(1, 3), (2, 4)	(3, 1), (2, 4)	−2	0
(1, 3), (2, 5)	(3, 1), (2, 5)	−2	0
(1, 4), (2, 5)	(3, 4), (2, 5)	—	—
(1, 4), (3, 5)	(3, 4), (1, 5)	—	—
(2, 4), (3, 5)	(2, 4), (1, 5)	—	—

Figure 6.5: Parity and links for  $E_2$

Right	Left	Parity	Link
(1, 2), (3, 4)	(5, 2), (3, 4)	0	0
(1, 2), (3, 5)	(5, 2), (3, 1)	+	—
(1, 2), (4, 5)	(5, 2), (4, 1)	+	—
(1, 3), (4, 5)	(5, 3), (4, 1)	+	—
(1, 4), (2, 3)	(5, 4), (2, 3)	0	0
(1, 5), (2, 3)	(5, 1), (2, 3)	0	0
(1, 5), (2, 4)	(5, 1), (2, 4)	0	0
(1, 5), (3, 4)	(5, 1), (3, 4)	0	0
(2, 3), (4, 5)	(2, 3), (4, 1)	0	0
(2, 5), (3, 4)	(2, 1), (3, 4)	0	0
(1, 3), (2, 4)	(5, 3), (2, 4)	−2	0
(1, 3), (2, 5)	(5, 3), (2, 1)	—	—
(1, 4), (2, 5)	(5, 4), (2, 1)	—	—
(1, 4), (3, 5)	(5, 4), (3, 1)	—	—
(2, 4), (3, 5)	(2, 4), (3, 1)	−2	0

Figure 6.6: Parity and links for  $E_3$

Right	Left	Parity	Link
(1, 2), (3, 4)	(3, 2), (5, 4)	0	0
(1, 2), (3, 5)	(3, 2), (5, 1)	0	0
(1, 2), (4, 5)	(3, 2), (4, 1)	0	0
(1, 3), (4, 5)	(3, 5), (4, 1)	—	—
(1, 4), (2, 3)	(3, 4), (2, 5)	0	0
(1, 5), (2, 3)	(3, 1), (2, 5)	—	—
(1, 5), (2, 4)	(3, 1), (2, 4)	—	—
(1, 5), (3, 4)	(3, 1), (5, 4)	0	0
(2, 3), (4, 5)	(2, 5), (4, 1)	—	—
(2, 5), (3, 4)	(2, 1), (5, 4)	0	0
(1, 3), (2, 4)	(3, 5), (2, 4)	0	0
(1, 3), (2, 5)	(3, 5), (2, 1)	—	—
(1, 4), (2, 5)	(3, 4), (2, 1)	—	—
(1, 4), (3, 5)	(3, 4), (5, 1)	—	—
(2, 4), (3, 5)	(2, 4), (5, 1)	—	—

Figure 6.7: Parity and links for  $E_4$

Right	Left	Parity	Link
(1, 2), (3, 4)	(5, 2), (1, 4)	—	—
(1, 2), (3, 5)	(5, 2), (1, 3)	—	—
(1, 2), (4, 5)	(5, 2), (4, 3)	0	0
(1, 3), (4, 5)	(5, 1), (4, 3)	0	0
(1, 4), (2, 3)	(5, 4), (2, 1)	0	0
(1, 5), (2, 3)	(5, 3), (2, 1)	0	0
(1, 5), (2, 4)	(5, 3), (2, 4)	—	—
(1, 5), (3, 4)	(5, 3), (1, 4)	—	—
(2, 3), (4, 5)	(2, 1), (4, 3)	0	0
(2, 5), (3, 4)	(2, 3), (1, 4)	0	0
(1, 3), (2, 4)	(5, 1), (2, 4)	—	—
(1, 3), (2, 5)	(5, 1), (2, 3)	—	—
(1, 4), (2, 5)	(5, 4), (2, 3)	—	—
(1, 4), (3, 5)	(5, 4), (1, 3)	—	—
(2, 4), (3, 5)	(2, 4), (1, 3)	0	0

Figure 6.8: Parity and links for  $E_5$



every two pseudopairs with non-planar final twists with the same parity have the same link.

*Proof.* We proceed by induction on the number of scales of  $G$ . If there is just 1, then the statement is trivial.

Suppose the statement is true for graphs with  $i$  scales,  $H_0, \dots, H_{i-1}$ . We consider the scale  $H_i$  with transition  $T_i$ . Let  $H$  be  $G$  restricted to the first  $i$  scales. Then applying induction to  $H$  gives a choice of embeddings for the first  $i - 1$  transitions. If the link of every pseudopair with non-planar final twist is the same as the parity of that twist, embed  $T_i$  as the appropriate  $E_i$ . Otherwise, choose  $\overline{E}_i$ . We proceed with the proof in the case where we choose  $E_i$  and note that the argument is identical in the other case.

We now consider a pseudopair using  $T_i, (P_1, P_2)$ . The restriction of  $(P_1, P_2)$  to  $H$  satisfies the properties of the lemma. Suppose that restriction had a planar final twist. Then its link is 0. If  $(P_1, P_2)$  also has planar final twist, then, consulting Figures 6.4 through 6.8 shows that the link is 0. If it does not have planar final twist, if  $T_i$  is of type 1, 2, or 3, then it now has link  $-1$  and parity  $+1$ . Otherwise it now has link  $+1$  and parity  $+1$ .

Suppose instead that the restriction had a non-planar final twist. Then it had parity  $+1$  and link  $+1$  or parity  $-1$  and link  $-1$ . We may assume by symmetry that it has parity  $+1$  and link  $+1$  and consult the above table. If  $(P_1, P_2)$  has a planar final twist, then the link of this transition is  $-1$  (again consulting Figures 6.4 through 6.8). Otherwise, if  $T_i$  is of type 1, 2, or 3, then the parity switches but the link stays the same, so it now has link  $+1$  and parity  $-1$ . Otherwise, the parity and link stay the same, so the final twist has link  $+1$  and parity  $+1$ .  $\square$

**Corollary 6.2.5.** *Snakelets are flat.*

*Proof.* Let  $C_1$  and  $C_2$  be vertex-disjoint cycles. By Lemma 6.2.1, we may assume  $C_1$  and  $C_2$  are lean. Then by Lemma 6.2.2 there is a pseudopair with planar final twist

whose link is the same as the linking number of  $C_1$  and  $C_2$ . By Lemma 6.2.4,  $C_1$  and  $C_2$  then have linking number 0. Since every pair of vertex disjoint cycles has 0 linking number, the graph is linkless and therefore flat.  $\square$

We would now like to extend the previous theorem to serpents. We discuss first tails of type 1. Let  $P$  be a tail of type 1 and let  $H_0$  be the scale of  $G$  with a transition to  $P$ . Let  $v_1, \dots, v_5$  be vertices on the outer cycle of  $H$  in order, such that there is a path on the outer cycle of  $H$  between  $v_1$  and  $v_5$  disjoint from a transition of  $H$ . Then there is a vertex of  $P$ ,  $u_1$  and 4 vertices in order on the outer cycle of  $P$ ,  $u_2, u_3, u_4, u_5$  such that there is a matching between  $\{u_1, u_3, u_5\}$  and  $\{v_1, v_3, v_5\}$  and a matching between  $\{u_2, u_4\}$  and  $\{v_2, v_4\}$ . It will be more convenient to assume that this matching is as simple as possible; specifically, we would like to assume that the edges of the matching are  $u_1v_1, u_3v_3, u_5v_5, u_2v_2, u_4v_4$ . We call such a tail, a *simple tail of type 1*.

**Lemma 6.2.6.** *Let  $G$  be a serpent with a tail of type 1,  $P$ , adjacent to a scale  $H_0$ . Then there is a graph  $G'$  obtained from  $G$  by replacing the edges between  $H_0$  and  $P$  by a scale  $H$  with transitions to  $P$  and  $H_0$  such that  $G$  is linkless if  $G'$  is linkless and, in  $G'$ ,  $P$  is a simple tail of type 1.*

*Proof.* Let  $v_1, \dots, v_5 \in V(H_0)$  and  $u_1 \dots u_5 \in V(P)$  be as defined above. Let  $T$  be the matching between them and let  $f$  be a function from  $v_1 \dots v_5$  onto  $\{1, 2, 3, 4, 5\}$  with  $f(v_i) = j$  if the edge  $v_iu_j \in T$ .

Let  $H$  be the cycle on 10 vertices  $a_1, \dots, a_{10}$  and chords  $a_1a_{10}, a_2a_9, a_3a_8, a_4a_7, a_5a_6$ . Remove the edges of  $T$  from  $G$  and add  $H$  to  $G$  along with the edges  $a_{10}u_1, a_9u_2, a_8u_3, a_7u_4, a_6u_5, v_1a_{f(v_1)}, v_2a_{f(v_2)}, v_3a_{f(v_3)}, v_4a_{f(v_4)}, v_5a_{f(v_5)}$  to get  $G'$ . Then  $G$  is clearly a minor of  $G'$ , so is linkless if  $G'$  is and  $P$  is a simple tail of type 1 in  $G'$ .  $\square$

We note that an embedding of  $G'$  in the previous lemma of the type we discuss in this paper immediately gives an embedding for  $G$  in the natural way.

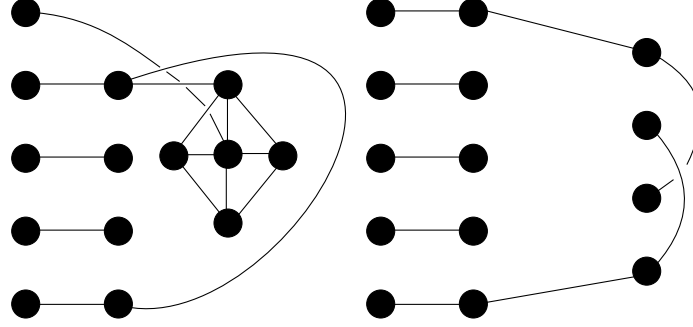


Figure 6.9: The two tails

For the remainder of the paper, we assume that tails of type 1 are simple. For each of the two types of tails, we give an embedding,  $S_1$  or  $S_2$  shown in Figure 6.9. Note that in  $S_1$ , the edge from  $v_1$  to  $u_1$  goes under every other edge in that tail. We define  $\overline{S}_1, \overline{S}_2$  analogously to  $\overline{E}_1, \dots, \overline{E}_5$ .

**Lemma 6.2.7.** *Let  $G$  be an embedded serpent with scales  $H_1, \dots, H_n$  and tails  $H_0$  and  $H_{n+1}$ . Suppose  $H_0$  is a simple tail of type 1 and embedded according to embedding  $S_1$  or  $\overline{S}_1$ . Then there is a serpent  $G'$  obtained from  $G$  by replacing  $H_0$  with a tail of type 2 embedded according to  $S_2$  or  $\overline{S}_2$  respectively such that the following is true. Let  $C_1$  and  $C_2$  be vertex disjoint cycles in  $G$ . Then there exist vertex disjoint cycles  $C'_1$  and  $C'_2$  in  $G'$  such that  $C'_1$  and  $C'_2$  have the same linking number as  $C_1$  and  $C_2$ .*

*Proof.* Let  $u_2, u_3, u_4, u_5$  be the four vertices in order on the outer cycle of  $H_0$  adjacent to vertices of  $H_1$  and let  $u_1$  be the other vertex of  $H_0$  adjacent to a vertex of  $H_1$ . We may assume the neighbors of  $v_i$  are  $u_i$ ,  $i \in \{1, 2, 3, 4, 5\}$  as in the definition of a simple tail of type 1. Let the edges between  $H_0$  and  $H_1$  be  $T$ .

Let  $H$  be the cycle on 9 vertices  $a_1, \dots, a_9$  with chords  $a_1a_9, a_2a_8, a_2a_9, a_3a_7, a_3a_8, a_4a_6, a_4a_7, a_5a_6, a_7a_9, a_6a_8$ . Replace  $H_0$  by  $H$  and add the edges  $v_i a_i, i \in \{1, 2, 3, 4, 5\}$  to find  $G'$ . Let  $T'$  be these added edges.  $G'$  is then a serpent which we embed as in the statement of the lemma.

Let  $C_1$  and  $C_2$  be vertex-disjoint cycles of  $G$ . We may assume  $C_1$  and  $C_2$  are lean

by Lemma 6.2.1. If  $C_1$  and  $C_2$  do not both use edges of  $T$ , then it is easy to find cycles in  $G'$  that agree with  $C_1$  and  $C_2$  on  $G \setminus H_0$  and then replace the subpath in  $H_0$  with a path in the outer cycle of  $H$ . So we may assume  $C_1$  and  $C_2$  each use two edges of  $T$ . Further, the contribution to the linking number in  $T \cup H_0$  must be non-zero, since otherwise it is easy to find  $C'_1$  and  $C'_2$  as desired. If neither uses the  $v_1u_1$  edge, then there is no contribution to the linking number in  $H_0 \cup T$ . So we may assume that  $C_1$  uses the edge  $v_1u_1$ . If  $C_1$  also uses the edge  $v_2u_2$  or  $v_5u_5$ , then again there is no contribution to the linking number in  $H_0$ , so we are done. So we may assume that  $C_1$  uses the  $v_3u_3$  edge or the  $v_4u_4$  edge. Let  $P_1$  be the subpath of  $C_1$  in  $H_0 \cup T$  with ends among the  $v_i, i \in \{1, 2, 3, 4, 5\}$  and  $P_2$  defined similarly for  $C_2$ .

Suppose  $C_1$  uses the  $v_3u_3$  edge. Then  $C_2$  must use the  $v_2u_2$  edge. If it uses the  $v_4u_4$  edge or the  $v_5u_5$  edge, replace  $P_1$  by  $v_1a_1a_9a_7a_3v_3$  and  $P_2$  by, respectively,  $v_2a_2a_8a_6a_4v_4$  or  $v_2a_2a_8a_6a_5v_5$  to find the desired cycles. If  $C_1$  uses the  $v_4u_4$  edge, then  $C_2$  uses the  $v_5u_5$  edge and one of the edges  $v_2u_2$  or  $v_3u_3$ . Then replace  $P_1$  by  $v_1a_1a_9a_7a_4$  and  $P_2$  by, respectively,  $v_5a_5a_6a_8a_2$  or  $v_5a_5a_6a_8a_3$  to find the desired cycles.  $\square$

**Lemma 6.2.8.** *Let  $G$  be an embedded serpent with scales  $H_1, \dots, H_n$  and tails  $H_0$  and  $H_{n+1}$ . Suppose  $H_0$  is of type 2 and embedded according to embedding  $S_2$  or  $\overline{S_2}$ . Then there is a serpent  $G'$  obtained from  $G$  by replacing  $H_0$  with two scales with a transition between them of type 2 embedded according to  $E_2$  or  $\overline{E_2}$  respectively such that the following is true. Let  $C_1$  and  $C_2$  be vertex disjoint cycles in  $G$ . Then there exist vertex disjoint cycles  $C'_1$  and  $C'_2$  in  $G'$  such that  $C'_1$  and  $C'_2$  have the same linking number as  $C_1$  and  $C_2$ .*

*Proof.* Let  $v_6, v_7, v_8, v_9$  be the four vertices of the outer cycle of  $H_0$  in order with edges  $v_6v_8$  and  $v_7v_9$  given by the definition of a tail of type 2. We define  $A_1$  to be  $H_0$  without the edges  $v_6v_8$  or  $v_7v_9$  and an additional vertex  $v_{10}$  added by subdividing the path in the outer cycle between  $v_8$  and  $v_9$  once. Let  $A_2$  be the cycle on ten vertices  $u_1, \dots, u_{10}$  with an chord between  $v_1$  and  $v_5$ . Let  $A$  be formed from  $A_1 \cup A_2$  by adding

edges  $v_6u_3, v_{10}u_2, v_7u_1, v_8u_4, v_9u_5$ . Then  $G'$  is formed from  $G$  by replacing  $H_0$  with  $A$ . Note that the edges we added between  $A_1$  and  $A_2$  form exactly a transition of type 2,  $T$ . We embed  $T$  according to  $E_2$  or  $\overline{E_2}$  as in the statement of the lemma.

We may assume  $H_0$  was embedded according to  $S_2$  by symmetry, so  $T$  is embedded according to  $E_2$ . Let  $C_1$  and  $C_2$  be vertex-disjoint cycles of  $G$ . If  $C_1$  and  $C_2$  do not contain  $v_6v_8$  or  $v_7v_9$ , then an image of them exists in  $G'$  embedded in the same way, so the corresponding cycles have the same linking number. If one of the cycles contains the edge  $v_6v_8$ , replace it by the path  $v_6u_3u_4v_8$  in  $G'$  and similarly if one of the cycles contains the path  $v_7v_9$  replace it by the path  $v_7u_1u_5v_9$ . The two cycles formed by these replacements are vertex disjoint and have the same linking number as  $C_1$  and  $C_2$  by our choice of embeddings.  $\square$

These lemmas lead to the following theorem:

**Lemma 6.2.9.** *Let  $G$  be a serpent. Then there is a choice of embeddings for the two tails chosen from  $S_i$  or  $\overline{S}_i$  for  $1 \leq i \leq 2$  and for each of the transitions of  $G$ ,  $T_1, \dots, T_n$ , chosen from either  $E_i$  or  $\overline{E}_i$  for the appropriate  $i$  as shown above such that any two vertex-disjoint cycles in  $G$  has linking number 0.*

*Proof.* If  $G$  has any tails of type 1, we can find a new graph  $G'$  whose embedding will tell us the appropriate embedding for  $G$  but which has no tails of type 1 by applying Lemma 6.2.7 at most twice. Then, if  $G'$  has any tails of type 2, we can find a new graph  $G''$  with no tails by applying Lemma 6.2.8 at most twice. We may then apply Lemma 6.2.4 to find the appropriate embedding.  $\square$

This immediately proves the main theorem of this chapter:

**Theorem 6.2.10.** *Serpents are flat.*

## CHAPTER VII

### CONCLUSION

In this chapter, we reiterate the main results of this thesis and discuss avenues of further research.

#### 7.1 *Coloring*

In Chapter 2, we proved as Theorem 2.1.4 the following:

**Theorem 7.1.1.** *Every planar graph without cycles of length four through six or eared seven cycles is 3-colorable.*

The desired extension, is, naturally, Steinberg's Conjecture, but there are several intermediate results that are of interest. One possible next step would be to consider planar graphs without cycles of length four or five or eared cycles of length six or seven. In this case, six cycles are generally easier to handle than seven cycles for the purpose of reducibility. By this we mean that if  $C$  is a 6-cycle with vertices  $v_1, v_2, v_3, v_4, v_5, v_6$  in order, then we can identify, for instance  $v_1$  with  $v_5$  and  $v_2$  with  $v_4$ . This, along with the two symmetric operations, immediately gives three smaller graphs, any of whose coloring extends to the bigger one. In addition, we can identify  $v_1, v_3, v_5$  to find another two potential reductions. There is one significant difficulty in this extension, however. In the case of Theorem 2.1.4, there was only one exceptional graph,  $H^*$  (the graph formed by taking an 11-cycle  $v_1, \dots, v_{11}$  and adding a vertex adjacent to  $v_1, v_2$ , and  $v_7$ ), exceptional in the sense that there are colorings of its outer cycle that do not extend to colorings of the entire graph. If we continue to allow an outer cycle of length 11, there are many exceptional graphs; however if we limit it to size nine, there are only two immediate small examples shown in Figure

7.1. We conjecture that these are, in fact, the only two exceptional graphs with outer cycle of length at most nine. Note that reducing the size of the outer cycle makes the reductions in Chapter 2 more difficult, but this remains a potentially fruitful avenue of investigation.

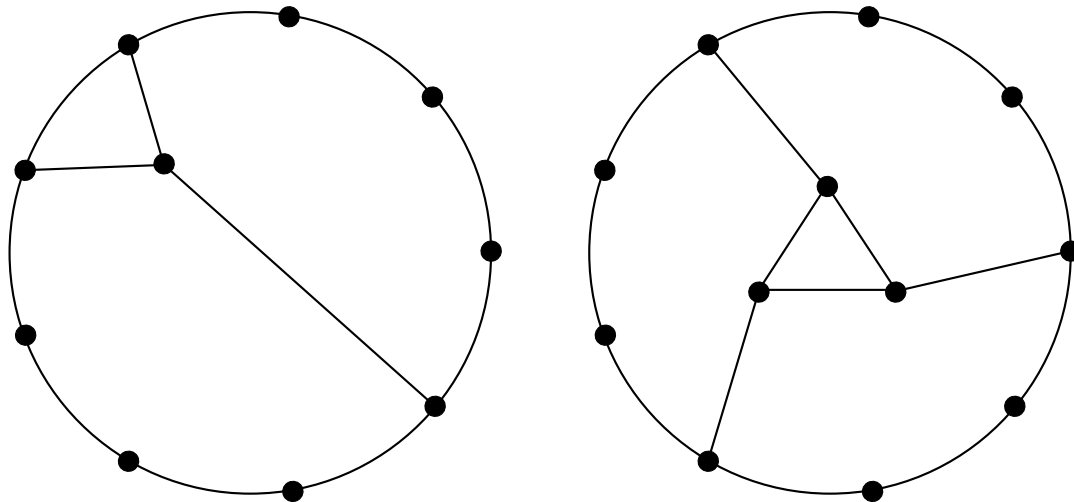


Figure 7.1: Exceptional graphs allowing six cycles

The above difficulty reflects the general problem with this technique in attempting Steinberg's Conjecture. The result of Borodin et al in [10] that planar graphs excluding cycles of length four through seven are three colorable contended with no exceptional graphs. The result in Chapter 2 handled exactly one. To handle the weakening of Steinberg's Conjecture restricted to just planar graphs that exclude cycles of length four through six, we would need to deal with infinitely many, even if we only extend from an outer cycle of length at most nine. For example, we consider the following construction. Let  $H_0$  be the graph depicted in Figure 7.2: it consists of an outer cycle of length 9 with vertices  $v_1, \dots, v_9$ , a triangle  $u_1, u_2, u_3$  disjoint from the outer cycle, and three additional vertices  $x_1, x_2, x_3$  along with edges  $v_1x_1, v_2x_1, x_1u_1, v_4x_2, v_5x_2, x_2u_2, v_7x_3, v_8x_3, x_3u_3$ . We note that  $H_0$  has three vertices on the outer cycle of degree 2; such will be the case for each graph  $H_i$ . Let  $H_i$  be the graph formed

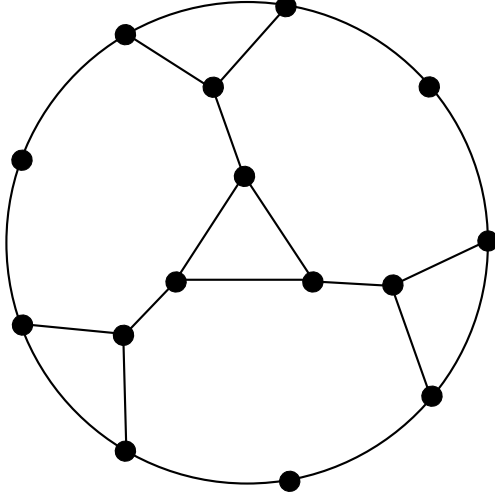


Figure 7.2: The graph  $H_0$

from  $H_{i-1}$  by adding nine vertices in a cycle,  $v_1, \dots, v_9$  and edges  $v_1u_1, v_2u_1, v_4u_2, v_5u_2, v_7u_3, v_8u_4$  where  $u_1, u_2, u_3$  are the vertices of degree 2 in  $H_{i-1}$ . Then note that  $v_3, v_6$  and  $v_9$  all have degree 2. We show  $H_1$  in Figure 7.3 as an example.

**Theorem 7.1.2.** *There is no proper 3-coloring of  $H_i$ ,  $i \geq 0$  in which the vertices of the outer cycle of degree 2 are colored the same.*

*Proof.* We proceed by induction. We use the notation above, that  $v_1, \dots, v_9$  form the outer cycle with vertices  $v_3, v_6, v_9$  of degree two,  $u_1, u_2, u_3$  the triangle with no neighbors on the outer face and  $x_1, x_2, x_3$  the final three vertices with  $x_1$  adjacent to  $u_1, v_1, v_2$ ,  $x_2$  adjacent to  $u_2, v_4, v_5$ , and  $x_3$  adjacent to  $u_3, v_7, v_8$ . Then if we color  $v_3, v_6, v_9$  all with color 1, then each of  $x_1, x_2, x_3$  must also be colored 1 which makes it impossible to color  $u_1, u_2$ , and  $u_3$ .

Assume the induction hypothesis holds for  $0 \leq i < n$ . Then for  $H_n$ , let  $v_1 \dots v_9$  be the vertices of the outer cycle with  $v_3, v_6, v_9$  of degree two and  $u_1, u_2, u_3$  the vertices not on the outer cycle with neighbors on the outer cycle. Then if we color  $v_3, v_6, v_9$  with color 1, then  $u_1, u_2, u_3$  are colored 1 as well, so by the induction hypothesis the resulting graph does not admit a proper 3-coloring.  $\square$



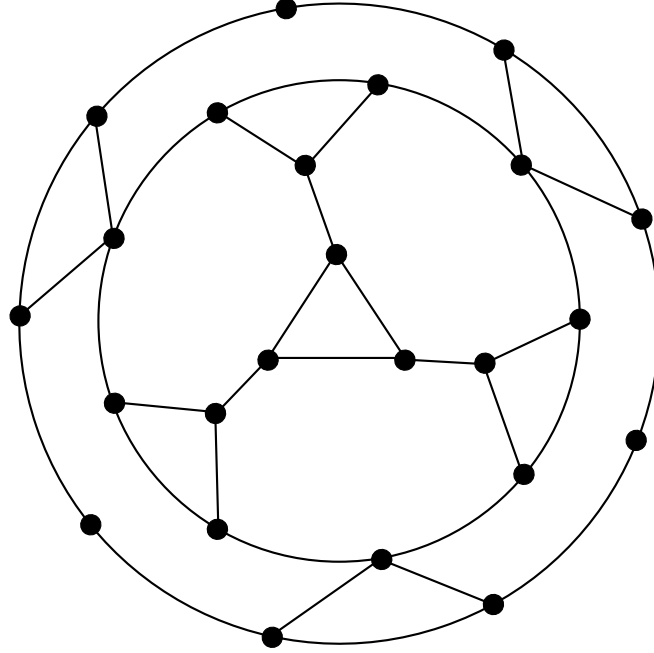


Figure 7.3: The graph  $H_1$

The graphs  $H_i, i \geq 0$  are not the only exceptional graphs (for another example, take a 9-cycle,  $v_1, \dots, v_9$  and add two adjacent vertices  $u_1, u_2$  and edges  $v_1u_1, v_2u_1, v_6u_2, v_7u_2$ ). If we wish to pursue a similar discharging-type argument a reasonable avenue to explore is to enumerate the small exceptional graphs and see if the larger ones can then be classified into particular families, like the  $H_i$  above. Clearly if we knew exactly all the exceptional graphs, we would have a proof of the conjecture, but if we could understand the bad colorings for the more obvious families we might be able to use that and a similar extension-type argument to find the others.

## 7.2 Pfaffian Orientations

In Chapter 3, we proved as Theorem 3.1.3 that

**Theorem 7.2.1.** *Every internally 4-connected bipartite non-planar graph has an odd hex.*

While we make use of this theorem in the context of Pfaffian orientations, this

theorem is nice in another sense. By a variation on Kuratowski's Theorem, we know that 3-connected non-planar bipartite graphs contain a subdivision of  $K_{3,3}$ . So we have a bipartite graph containing another bipartite graph as a subdivision. A reasonable question, then, is when can this be done in a bipartite way, in the sense that the subdivision is odd? There are a number of other properties that can be viewed in this way; for instance when does a non-outer planar bipartite graph contain an odd subdivision of  $K_{2,3}$ ? In the context of Pfaffian orientations, it would be nice to understand this odd subdivision containment for non-bipartite graphs. While being internally 4-connected is not a strong enough condition to guarantee an odd  $K_{3,3}$  subdivision (take  $V_8$  as an example), a theorem in a similar vein would have useful applications to the general Pfaffian orientation question.

In Chapter 4, we reproved several theorems of [42]. In doing so, we developed on the techniques from Chapter 3. The main new area of research here is whether we can make use of these techniques in graphs that are not bipartite, particularly in bricks. We note that many of the tools that we used in this chapter, unlike the previous, made use of alternating paths rather than vertex parity, which leaves some hope that they could be extended to a more general class of graphs with matching characteristics. These, or similar, techniques, then could be useful in converting a structural theorem in the vein of Little's Theorem for bricks into a polynomial time algorithm.

One appropriate intermediate result could be to work in near-bipartite graphs:

**Definition.** Let  $G$  be a matching-covered graph such that there exists a pair of edges,  $e$  and  $f$  such that  $G - \{e, f\}$  is bipartite and matching covered. Then we say that  $G$  is *near-bipartite*.

For near-bipartite graphs, the question of finding a Pfaffian orientation has already been solved, but requires one further definition. We say that a graph is a *weak matching minor* of another if the first can be obtained from a matching minor of the

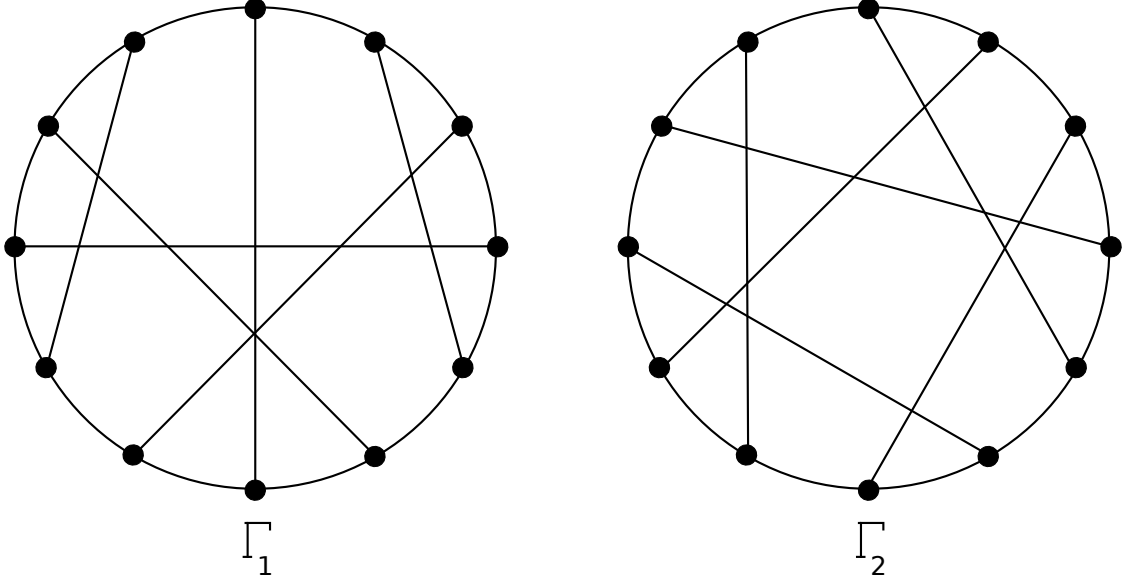


Figure 7.4: The graphs  $\Gamma_1$  and  $\Gamma_2$

other by contracting odd cycles and deleting all resulting loops and parallel edges. Let  $\Gamma_1$  and  $\Gamma_2$  be as shown in Figure 7.4. Then Fischer and Little [15] showed that

**Theorem 7.2.2.** *A near-bipartite graph admits a Pfaffian orientation if and only if it has no weak matching minor isomorphic to  $K_{3,3}$ ,  $\Gamma_1$ , or  $\Gamma_2$ .*

While this theorem does not give a polynomial-time characterization, that problem has been solved in [32]. It remains a natural question, then, to see whether our techniques can be used to prove a similar algorithm and whether they can be extended to more general classes of graphs.

### 7.3 Flat Embeddings

In Chapter 5, we proved as Theorem 5.2.1 a structural result for flat graphs:

**Theorem 7.3.1.** *There exists an absolute constant  $N$  such that if  $G$  is a graph on at least  $N$  vertices at least one of the following holds:*

- (1)  $G$  contains a graph in the Petersen family as a minor

- (2)  *$G$  contains a complete separation*
- (3)  *$G$  contains a peripheral theta, the deletion of whose arc leaves  $G$  Kuratowski-connected*
- (4) *There exists  $X \subseteq V(G)$ ,  $|X| \leq 1$  such that  $G \setminus X$  is planar*

One immediate question that arises is whether this theorem holds for all graphs, not just sufficiently large ones. This would be an interesting result, though its applications in our context are likely limited. That context is an attempt to extend this theorem to a polynomial-time algorithm to determine whether a graph is flat. The proof of the theorem itself is naturally algorithmic, but there are a number of technical difficulties in using it to find an algorithm to construct a flat embedding. The first difficulty is in bounding the complexity of that embedding, which leads to a natural question: Is there a representation of a flat embedding of a graph with polynomial complexity in the size of the graph? In this case, the standard notions of complexity of an embedding are either the number of crossings in the regular projection or the number of pieces in a piecewise linear embedding, though it's possible that there is some other representation of an embedding that is more appropriate for this purpose.

A natural algorithmic implementation of the above theorem is to, assuming outcomes two or three, break the graph into smaller pieces and embed those. A difficulty is in then putting the pieces back together. In either case, we need at least one panel on at least one cycle, but finding such a panel in an embedding is difficult if the embedding is not carefully constructed. In fact, if we had an algorithm to find a panel on a particular cycle in a fixed embedding, such an algorithm would immediately solve the long-open unknotting problem. But the embedding that we have is not arbitrary, so a plausible approach to this problem would be to find some invariant of the embedding or to keep track of panels on important cycles throughout the recursion.

Even failing to find the complete algorithm, there are several intermediate results

that could be of interest. For example, what if we ignore the difficulty of keeping track of cycles. Suppose we have a polynomial-time oracle to find a panel on a cycle in a flat embedding. Is there a polynomial-time algorithm for finding a flat embedding of a graph given such an oracle? Can we weaken such an oracle to only solving the unknotting problem? That is to say, is there a polynomial time algorithm for finding a flat embedding given a polynomial-time oracle for the unknotting problem?

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